

QUANTUM ISOMETRY GROUPS

JYOTISHMAN BHOWMICK



Indian Statistical Institute, Kolkata
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JYOTISHMAN BHOWMICK

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Thesis Advisor: Debashish Goswami

Indian Statistical Institute
203, B.T. Road, Kolkata, India.

In the memory of my grandmother

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Notations

\mathbb{N}	The set of natural numbers
\mathbb{N}_0	$\mathbb{N} \cup \{0\}$
\mathbb{R}	The set of real numbers
\mathbb{C}	The set of complex numbers
$\mathcal{M}_n(\mathbb{C})$	The set of all $n \times n$ complex matrices
S^1	The circle group
\mathbb{T}^n	The n -torus
ev	Evaluation map
id	The identity map
$\ell^2(\mathbb{N})$	The Hilbert space of square summable sequences
$C^\infty(M)$	The space of smooth functions on a smooth manifold M
$C_c^\infty(M)$	The space of compactly supported smooth functions on M
$V_1 \otimes_{\text{alg}} V_2$	Algebraic tensor product of two vector spaces V_1 and V_2
$\mathcal{A} \otimes \mathcal{B}$	Minimal tensor product of two C^* algebras \mathcal{A} and \mathcal{B}
$\mathcal{M}(\mathcal{A})$	The multiplier algebra of a C^* algebra \mathcal{A}
$\mathcal{L}(E, F)$	The space of adjointable maps from Hilbert modules E to F
$\mathcal{L}(E)$	The space of adjointable maps from a Hilbert module E to itself
$\mathcal{K}(E, F)$	The space of compact operators from Hilbert modules E to F
$\mathcal{K}(E)$	The space of compact operators from a Hilbert module E to itself
$\mathcal{B}(\mathcal{H})$	The set of all bounded linear operators on a Hilbert space \mathcal{H}
$\mathcal{A} * \mathcal{B}$	Free product of two C^* algebras \mathcal{A} and \mathcal{B}
$G * H$	Free product of two groups G and H
$G \rtimes H$	Semi direct product of two groups G and H

Chapter 0

Introduction

The theme of this thesis lies on the interface of two areas of the so called “ noncommutative mathematics ”, namely noncommutative geometry (NCG) a la Connes, cf [17] and the theory of (C^* -algebraic) compact quantum groups (CQG) a la Woronowicz, cf [67] which are generalizations of classical Riemannian spin geometry and that of compact topological groups respectively.

The root of NCG can be traced back to the Gelfand Naimark theorem which says that there is an anti-equivalence between the category of (locally) compact Hausdorff spaces and (proper, vanishing at infinity) continuous maps and the category of (not necessarily) unital C^* algebras and $*$ -homomorphisms. This means that the entire topological information of a locally compact Hausdorff space is encoded in the commutative C^* algebra of continuous functions vanishing at infinity. This motivates one to view a possibly noncommutative C^* algebra as the algebra of “functions on some noncommutative space”.

In classical Riemannian geometry on spin manifolds, the Dirac operator on the Hilbert space $L^2(S)$ of square integrable sections of the spinor bundle contains a lot of geometric information. For example, the metric, the volume form and the dimension of the manifold can be captured from the Dirac operator. This motivated Alain Connes to define his noncommutative geometry with the central object as the spectral triple which is a triplet $(\mathcal{A}, \mathcal{H}, D)$ where \mathcal{H} is a separable Hilbert space, \mathcal{A} is a (not necessarily closed) $*$ -algebra of $\mathcal{B}(\mathcal{H})$, D is a self adjoint (typically unbounded) operator (sometimes called the Dirac operator of the spectral triple) such that $[D, a]$ admits a bounded extension. This generalizes the classical spectral triple $(C^\infty(M), L^2(S), D)$ on any Riemannian spin manifold M , where D denotes the usual Dirac operator.

On the other hand, quantum groups have their origin in different problems in mathematical physics as well as the theory of classical locally compact groups. It was S.L. Woronowicz, who in [66] and [67] was able to pinpoint a set of axioms for defining

compact quantum groups (CQG for short) as the correct generalization of compact topological groups.

The idea of a group acting on a space was extended to the idea of a CQG co-acting on a noncommutative space (that is, a possibly noncommutative C^* algebra). Following suggestions of Alain Connes, Shuzhou Wang in [60] defined and proved the existence of quantum automorphism groups on finite dimensional C^* algebras. Since then, many interesting examples of such quantum groups, particularly the quantum permutation groups of finite sets and finite graphs, were extensively studied by a number of mathematicians (see for example [1], [2], [3], [4], [5], [61] and references therein). The underlying basic principle of defining a quantum automorphism group corresponding to some given mathematical structure (for example a finite set, a graph, a C^* or von Neumann algebra) consists of two steps: first, to identify (if possible) the group of automorphism of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category, replacing groups by quantum groups of appropriate type. However, most of the work done by them concerned some kind of quantum automorphism group of a ‘finite’ structure, for example, of finite sets or finite dimensional matrix algebras. It was thus quite natural to try to extend these ideas to the ‘infinite’ or ‘continuous’ mathematical structures, for example classical and noncommutative manifolds. With this motivation, Goswami ([30]) formulated and studied the quantum analogues of the group of Riemannian isometries called the quantum isometry group. Classically, an isometry is characterized by the fact that its action commutes with the Laplacian. Therefore, to define the quantum isometry group, it is reasonable to consider a category of compact quantum groups which act on the manifold (or more generally on a noncommutative manifold given by a spectral triple) with the action commuting with the Laplacian, say \mathcal{L} , coming from the spectral triple. It is proven in [30] that a universal object in the category (denoted by $\mathbf{Q}'_{\mathcal{L}}$) of such quantum groups does exist (denoted by $QISO^{\mathcal{L}}$) if one makes some mild assumptions on the spectral triple all of which are valid for a compact Riemannian spin manifold. The work of this thesis starts with the computation of the quantum isometry group in several commutative and noncommutative examples ([6], [7]).

However, the formulation of quantum isometry groups in [30] had a major drawback from the view point of noncommutative geometry, since it needed a ‘good’ Laplacian to exist. In noncommutative geometry it is not always easy to verify such an assumption about the Laplacian, and thus it would be more appropriate to have a formulation in terms of the Dirac operator directly. This is what is done in [8] where the notion of a quantum group analogue of the group of orientation preserving isometries was given and its existence as the universal object in a suitable category was proved. Then, a number of computations for this quantum group were done in [8], [9] and [10].

Now we try to give an idea of the contents of each of the chapters. In chapter 1, we discuss the concepts and results needed in the later chapters of the thesis. For the sake of completeness, we begin with a glimpse of operator algebras and Hilbert modules, free product and tensor products of C^* algebras and some examples. The next section is on quantum groups which we start with the basics of Hopf algebras and then define compact quantum groups (CQG) and give relevant definitions and properties including a brief review of Peter Weyl theory. After that, we introduce the quantum groups $U_\mu(2)$, $SU_\mu(2)$, $A_u(Q)$ and $\mathcal{U}_\mu(su(2))$. We end this section by introducing the notion of a C^* action of a compact quantum group on a C^* algebra and giving an account of Shuzhou Wang's work in [60]. The next section is on Rieffel deformation where we recall a part of the work done in [46] and [62] which are relevant to us. We describe some important examples which are going to appear in chapter 4. The 4th section is on classical Riemannian geometry which includes, among other things, the definition and properties of Dirac operator which will serve as a motivation for the definition of 'spectral triple' in the 5th section. This section also contains a subsection on isometry groups of classical Riemannian manifold in which the characterizing property of an isometry, in the form given in [30], is given. In the 5th section, we define spectral triples, give examples of them, introduce the Hilbert space of forms (as in [29]), noncommutative volume form and the notion of Laplacian in noncommutative geometry.

Chapter 2 is on the Laplacian-based approach to quantum isometry groups as proposed in [30]. Here we recall the formulation of quantum isometry groups from [30] and then compute them for the space of continuous functions on the classical 2-spheres, the circle and the n -tori. In each of these cases, the quantum isometry group turns out to be the same as the classical ones, that is, $C(O(3))$, $C(S^1 \rtimes_{\mathbb{Z}_2})$ and $C(\mathbb{T}^n \rtimes_{\mathbb{Z}_2^n \rtimes S_n})$ respectively (S_n being the permutation group on n symbols).

Chapter 3 deals with the quantum group of orientation preserving isometries. The classical situation is stated clearly, which will serve as a motivation for the quantum formulation. Then, the quantum group of orientation-preserving isometries of an R -twisted spectral triple is defined (see [31] for the definition of an R -twisted spectral triple) and its existence is proven. Given an R -twisted spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ of compact type, we consider a category \mathbf{Q}' of pairs (Q, U) where Q is a compact quantum group which has a unitary (co)-representation U on \mathcal{H} commuting with D , and such that for all state ϕ on Q , $(\text{id} \otimes \phi)ad_U$ maps \mathcal{A}^∞ inside \mathcal{A}''_∞ . Moreover, let \mathbf{Q}'_R be a subcategory of \mathbf{Q}' consisting of those (Q, U) for which ad_U preserves the R -twisted volume form. In section 3.2, we have proved that \mathbf{Q}'_R has a universal object to be denoted by $\widetilde{QISO}_R^+(D)$. The Woronowicz C^* subalgebra of $\widetilde{QISO}_R^+(D)$ generated by elements of the form $\langle ad_U(a)(\eta \otimes 1), \eta' \otimes 1 \rangle_{\widetilde{QISO}_R^+(D)}$ where η, η' are in \mathcal{H} , a is in \mathcal{A}^∞

and $\langle \cdot, \cdot \rangle_{\widetilde{QISO_R^+(D)}}$ denotes the $\widetilde{QISO_R^+(D)}$ valued inner product of $\mathcal{H} \otimes \widetilde{QISO_R^+(D)}$ is defined to be the quantum group of orientation and volume preserving isometries of the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ and is denoted by $QISO_R^+(D)$. The next section explores the conditions under which the action of this compact quantum group keeps the C^* algebra invariant and is a C^* action. Moreover, we have given some sufficient conditions under which the universal object in the bigger category \mathbf{Q}' exists which is denoted by $\widetilde{QISO^+(D)}$ and the corresponding Woronowicz C^* subalgebra as above is denoted by $QISO^+(D)$. After this, we compare this approach with the Laplacian-based approach in [30]. We obtain the following results:

(1) Under some reasonable conditions, $QISO_I^+(D)$ is a sub-object of $QISO^\mathcal{L}$ in the category $\mathbf{Q}'_\mathcal{L}$.

(2) $QISO^\mathcal{L}$ is isomorphic to a quantum subgroup of $QISO_I^+$ corresponding to the Hodge Dirac operator coming from D .

(3) Moreover, under some conditions which are valid for compact spin manifolds, $QISO^\mathcal{L}$ and the $QISO_I^+$ of the Hodge Dirac operator are isomorphic.

The next section is on examples and computations. To begin with, the compact quantum group $U_\mu(2)$ is identified as the $QISO^+$ of $SU_\mu(2)$ corresponding to the spectral triple constructed by Chakraborty and Pal in [13]. Then we derive that $QISO^+$ for the classical spectral triple on $C(\mathbb{T}^2)$ is $C(\mathbb{T}^2)$ itself. We end the chapter by showing that $QISO^+$ of spectral triples associated with some approximately finite dimensional C^* algebras arise as the inductive limit of $QISO^+$ of the constituent finite dimensional algebras. The results of this chapter are taken from [8] and [9].

Chapter 4 is about the $QISO^\mathcal{L}$ and $QISO_R^+$ of a Rieffel deformed noncommutative manifold. We first discuss the isospectral deformation of a spectral triple, followed by the proof of some preparatory technical results which will be needed later. Then in the final section we prove that $QISO_R^+$ and $QISO^\mathcal{L}$ of a Rieffel deformed (noncommutative) manifold is a Rieffel deformation of the $QISO_R^+$ and $QISO^\mathcal{L}$ (respectively) of the original (undeformed) manifold.

In chapter 5, we compute the quantum group of orientation preserving isometries for two different families of spectral triples on the Podles spheres, one constructed by Dabrowski et al in [24] and the other by Chakraborty and Pal in [14]. We start by giving the different descriptions of the Podles spheres (as in [43], [24], [37], and [50]) and the formula for the Haar functional on it. Then we introduce the spectral triples on the Podles spheres as in [24] and show that it is indeed $SU_\mu(2)$ equivariant and R -twisted (for a suitable R). After this, the compact quantum group $SO_\mu(3)$ is defined and its action on the Podles sphere is discussed. In the 3rd section, the computation for identifying $SO_\mu(3)$ as $QISO_R^+$ for this spectral triple is given. In the 4th section, the spectral triple defined in [14] is introduced and then the corresponding $QISO^+$

is computed. In particular, it follows that $QISO^+$ in general may not be a matrix quantum group and that it may not have a C^* action.

Chapter 1

Preliminaries

1.1 Operator algebras and Hilbert modules

We presume the reader's familiarity with the theory of operator algebras and Hilbert modules. However, for the sake of completeness, we give a sketchy review of some basic definitions and facts and refer to [56], [39] for the details. Throughout this thesis, all algebras will be over \mathbb{C} unless otherwise mentioned.

1.1.1 C^* algebras

A C^* algebra \mathcal{A} is a Banach $*$ -algebra satisfying the C^* property : $\|x^*x\| = \|x\|^2$ for all x in \mathcal{A} . The algebra \mathcal{A} is said to be unital or non-unital depending on whether it has an identity or not. Every commutative C^* algebra \mathcal{A} is isometrically isomorphic to the C^* algebra $C_0(X)$ consisting of complex valued functions on a locally compact Hausdorff space X vanishing at infinity (Gelfand's theorem). An arbitrary (possibly noncommutative) C^* algebra is isometrically isomorphic to a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$, the set of all bounded operators on a Hilbert space \mathcal{H} .

For x in \mathcal{A} , the **spectrum** of x , denoted by $\sigma(x)$ is defined as the complement of the set $\{z \in \mathbb{C} : (z1 - x)^{-1} \in \mathcal{A}\}$. An element x in \mathcal{A} is called **self adjoint** if $x = x^*$, **normal** if $x^*x = xx^*$, **unitary** if $x^* = x^{-1}$, **projection** if $x = x^* = x^2$ and **positive** if $x = y^*y$ for some y in \mathcal{A} . When x is normal, there is a continuous functional calculus sending f in $C(\sigma(x))$ to $f(x)$ in \mathcal{A} . where $f \mapsto f(x)$ is a $*$ isometric isomorphism from $C(\sigma(x))$ onto $C^*(x)$.

A linear map between two C^* algebras is said to be **positive** if it maps positive elements to positive elements. A positive linear functional ϕ such that $\phi(1) = 1$ is called a **state** on \mathcal{A} . A state ϕ is called a **trace** if $\phi(ab) = \phi(ba)$ for all a, b in \mathcal{A} and **faithful** if $\phi(x^*x) = 0$ implies $x = 0$. Given a state ϕ on a C^* algebra \mathcal{A} , there exists a triple (called the GNS triple) $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ consisting of a Hilbert space \mathcal{H}_ϕ , a $*$

representation π_ϕ of \mathcal{A} into $\mathcal{B}(\mathcal{H}_\phi)$ and a vector ξ_ϕ in \mathcal{H}_ϕ which is cyclic in the sense that $\{\pi_\phi(x)\xi_\phi : x \in \mathcal{A}\}$ is total in \mathcal{H}_ϕ satisfying

$$\phi(x) = \langle \xi_\phi, \pi_\phi(x)\xi_\phi \rangle.$$

For a two-sided norm closed ideal \mathcal{I} of a C^* algebra \mathcal{A} , the canonical quotient norm on the Banach space \mathcal{A}/\mathcal{I} is in fact the unique C^* norm making \mathcal{A}/\mathcal{I} into a C^* algebra. Here we prove two results which we are going to need later on.

Lemma 1.1.1. [30] *Let \mathcal{C} be a C^* algebra and \mathcal{F} be a nonempty collection of C^* -ideals (closed two-sided ideals) of \mathcal{C} . Then for any x in \mathcal{C} , we have*

$$\sup_{I \in \mathcal{F}} \|x + I\| = \|x + I_0\|,$$

where I_0 denotes the intersection of all I in \mathcal{F} and $\|x + I\| = \inf\{\|x - y\| : y \in I\}$ denotes the norm in \mathcal{C}/I .

Proof : It is clear that $\sup_{I \in \mathcal{F}} \|x + I\|$ defines a norm on \mathcal{C}/I_0 , which is in fact a C^* norm since each of the quotient norms $\| \cdot + I \|$ is so. Thus the lemma follows from the uniqueness of C^* norm on the C^* algebra \mathcal{C}/I_0 . \square

Lemma 1.1.2. *Let \mathcal{C} be a unital C^* algebra and \mathcal{F} be a nonempty collection of C^* -ideals (closed two-sided ideals) of \mathcal{C} . Let \mathcal{I}_0 denote the intersection of all \mathcal{I} in \mathcal{F} , and let $\rho_{\mathcal{I}}$ denote the map $\mathcal{C}/\mathcal{I}_0 \ni x + \mathcal{I}_0 \mapsto x + \mathcal{I} \in \mathcal{C}/\mathcal{I}$ for \mathcal{I} in \mathcal{F} . Denote by Ω the set $\{\omega \circ \rho_{\mathcal{I}}, \mathcal{I} \in \mathcal{F}, \omega \text{ state on } \mathcal{C}/\mathcal{I}\}$, and let K be the weak-* closure of the convex hull of $\Omega \cup (-\Omega)$. Then K coincides with the set of bounded linear functionals ω on $\mathcal{C}/\mathcal{I}_0$ satisfying $\|\omega\| = 1$ and $\omega(x^* + \mathcal{I}_0) = \overline{\omega(x + \mathcal{I}_0)}$.*

Proof : We will use Lemma 1.1.1. Clearly, K is a weak-* compact, convex subset of the unit ball $(\mathcal{C}/\mathcal{I}_0)_1^*$ of the dual of $\mathcal{C}/\mathcal{I}_0$, satisfying $-K = K$. If K is strictly smaller than the self-adjoint part of unit ball of the dual of $\mathcal{C}/\mathcal{I}_0$, we can find a state ω on $\mathcal{C}/\mathcal{I}_0$ which is not in K . Considering the real Banach space $X = (\mathcal{C}/\mathcal{I}_0)_{\text{s.a.}}^*$ and using standard separation theorems for real Banach spaces (for example, Theorem 3.4 of [49], page 58), we can find a self-adjoint element x of \mathcal{C} such that $\|x + \mathcal{I}_0\| = 1$, and

$$\sup_{\omega' \in K} \omega'(x + \mathcal{I}_0) < \omega(x + \mathcal{I}_0).$$

Let γ belonging to \mathbb{R} be such that $\sup_{\omega' \in K} \omega'(x + \mathcal{I}_0) < \gamma < \omega(x + \mathcal{I}_0)$. Fix $0 < \epsilon < \omega(x + \mathcal{I}_0) - \gamma$, and let \mathcal{I} be an element of \mathcal{F} be such that $\|x + \mathcal{I}_0\| - \frac{\epsilon}{2} \leq \|x + \mathcal{I}\| \leq \|x + \mathcal{I}_0\|$. Let ϕ be a state on \mathcal{C}/\mathcal{I} such that $\|x + \mathcal{I}\| = |\phi(x + \mathcal{I})|$. Since x is self-adjoint, either $\phi(x + \mathcal{I})$ or $-\phi(x + \mathcal{I})$ equals $\|x + \mathcal{I}\|$, and $\phi' := \pm \phi \circ \rho_{\mathcal{I}}$, where the sign is chosen so

that $\phi'(x + \mathcal{I}_0) = \|x + \mathcal{I}\|$. Thus, ϕ' is in K , so $\|x + \mathcal{I}_0\| = \phi'(x + \mathcal{I}) \leq \gamma < \omega(x + \mathcal{I}_0) - \epsilon$. But this implies $\|x + \mathcal{I}_0\| \leq \|x + \mathcal{I}\| + \frac{\epsilon}{2} < \omega(x + \mathcal{I}_0) - \frac{\epsilon}{2} \leq \|x + \mathcal{I}_0\| - \epsilon$ (since ω is a state), which is a contradiction completing the proof. \square

For a C^* algebra \mathcal{A} (possibly non unital), its multiplier algebra, denoted by $\mathcal{M}(\mathcal{A})$, is defined as the maximal C^* algebra which contains \mathcal{A} as an essential two sided ideal, that is, \mathcal{A} is an ideal in $\mathcal{M}(\mathcal{A})$ and for y in $\mathcal{M}(\mathcal{A})$, $ya = 0$ for all a in \mathcal{A} implies $y = 0$. The norm of $\mathcal{M}(\mathcal{A})$ is given by $\|x\| = \sup_{a \in \mathcal{A}, \|a\| \leq 1} \{\|xa\|, \|ax\|\}$. There is a locally convex topology called the strict topology on $\mathcal{M}(\mathcal{A})$, which is given by the family of seminorms $\{\|\cdot\|_a, a \in \mathcal{A}\}$, where $\|x\|_a = \text{Max}(\|xa\|, \|ax\|)$, for x in $\mathcal{M}(\mathcal{A})$. $\mathcal{M}(\mathcal{A})$ is the completion of \mathcal{A} in the strict topology.

We now come to the **inductive limit of C^* algebras**. Let I be a directed set and $\{\mathcal{A}_i\}_{i \in I}$ be a family of C^* algebras equipped with a family of C^* homomorphisms $\Phi_{ij} : \mathcal{A}_j \rightarrow \mathcal{A}_i$ (when $j < i$) such that $\Phi_{ij} = \Phi_{ik}\Phi_{kj}$ when $j < k < i$. Then there exists a unique C^* algebra denoted by $\lim_i \mathcal{A}_i$ and C^* homomorphisms $\phi_i : \mathcal{A}_i \rightarrow \lim_i \mathcal{A}_i$ with the universal property that given any other C^* algebra \mathcal{A}' and C^* homomorphisms $\psi_i : \mathcal{A}_i \rightarrow \mathcal{A}'$ satisfying $\psi_j = \psi_i \Phi_{ij}$ for $j < i$, then there exists unique C^* homomorphism $\chi : \lim_i \mathcal{A}_i \rightarrow \mathcal{A}'$ satisfying $\chi \phi_i = \psi_i$. $\lim_i \mathcal{A}_i$ is called the inductive limit C^* algebra corresponding to the inductive system $(\mathcal{A}_i, \Phi_{ij})$. Inductive limit of a sequence of finite dimensional C^* algebras are called approximately finite dimensional C^* algebras or AF algebras.

A large class of C^* algebras are obtained by the following construction. Let \mathcal{A}_0 be an associative $*$ -algebra without any a-priori norm such that the set $\mathcal{F} = \{\pi : \mathcal{A}_0 \rightarrow \mathcal{B}(\mathcal{H}_\pi) \text{ } * \text{-homomorphism, } \mathcal{H}_\pi \text{ a Hilbert space}\}$ is non empty and $\|\cdot\|_u$ defined by $\|a\|_u = \sup\{\|\pi(a)\| : \pi \in \mathcal{F}\}$ is finite for all a . Then the completion of \mathcal{A}_0 in $\|\cdot\|_u$ is a C^* algebra known as the universal C^* algebra corresponding to \mathcal{A}_0 .

Example 1: Noncommutative two-torus

Let θ belongs to $[0, 1]$. Consider the $*$ algebra \mathcal{A}_0 generated by two unitary symbols U and V satisfying the relation $UV = e^{2\pi i \theta} VU$. It has a representation π on the Hilbert space $L^2(S^1)$ defined by $\pi(U)(f)(z) = f(e^{2\pi i \theta} z)$, $\pi(V)(f)(z) = zf(z)$ where f is in $L^2(S^1)$, z is in S^1 . Then $\|a\|_u$ is finite for all a in \mathcal{A}_0 . The resulting C^* algebra is called noncommutative two-torus and denoted by \mathcal{A}_θ .

Example 2: Group C^* algebra

Let G be a locally compact group with left Haar measure μ . One can make $L^1(G)$

into a Banach $*$ -algebra by defining

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)d\mu(s),$$

$$f^*(t) = \Delta(t)^{-1} \overline{f(t^{-1})}.$$

Here f, g are in $L^1(G)$, Δ is the modular homomorphism of G .

$L^1(G)$ has a distinguished representation π_{reg} on $L^2(G)$ defined by $\pi_{reg}(f) = \int f(t)\pi(t)d\mu(t)$ where $\pi(t)$ is a unitary operator on $L^2(G)$ defined by $(\pi(t)f)(g) = f(t^{-1}g)$ ($f \in L^2(G)$, $t, g \in G$). The reduced group C^* algebra of G is defined to be $C_r^*(G) := \overline{\pi_{reg}(L^1(G))}^{\mathcal{B}(L^2(G))}$.

Remark 1.1.3. For G abelian, we have $C_r^*(G) \cong C_0(\widehat{G})$ where \widehat{G} is the group of characters on G .

One can also consider the universal C^* algebra described before corresponding to the Banach $*$ -algebra $L^1(G)$. This is called the free or full group C^* algebra and denoted by $C^*(G)$.

Remark 1.1.4. For the so-called amenable groups (which include compact and abelian groups) we have $C^*(G) \cong C_r^*(G)$.

1.1.2 von Neumann algebras

We recall that for a Hilbert space \mathcal{H} , the **strong operator topology** (**SOT**) , the **weak operator topology** (**WOT**) and the **ultra weak topology** are the locally convex topologies on $\mathcal{B}(\mathcal{H})$ given by families of seminorms $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ respectively where $\mathcal{F}_1 = \{p_\xi : \xi \in \mathcal{H}\}$, $\mathcal{F}_2 = \{p_{\xi, \eta} : \xi, \eta \in \mathcal{H}\}$, $\mathcal{F}_3 = \{p_\rho : \rho \text{ is a trace class operator on } \mathcal{H}\}$ and $p_\xi(x) = \|x\xi\|$, $p_{\xi, \eta}(x) = |\langle x\xi, \eta \rangle|$, $p_\rho(x) = |\text{Tr}(x\rho)|$ (where Tr denotes the usual trace on $\mathcal{B}(\mathcal{H})$).

Now we state a well known fact.

Lemma 1.1.5. If T_n is a sequence of bounded operators converging to zero in SOT, then for any trace class operator W , $\text{Tr}(T_n W) \rightarrow 0$ as $n \rightarrow \infty$.

For any subset \mathcal{B} of $\mathcal{B}(\mathcal{H})$, we denote by \mathcal{B}' the commutant of \mathcal{B} , that is, $\mathcal{B}' = \{x \in \mathcal{B}(\mathcal{H}) : xb = bx \text{ for all } b \in \mathcal{B}\}$. A unital C^* subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a **von Neumann algebra** if $\mathcal{A} = \mathcal{A}''$ which is equivalent to being closed in any of the three topologies mentioned above.

A state ϕ on a von Neumann algebra \mathcal{A} is called normal if $\phi(x_\alpha)$ increases to $\phi(x)$ whenever x_α increases to x . A state ϕ on \mathcal{A} is normal if and only if there is a trace class

operator ρ on \mathcal{H} such that $\phi(x) = \text{Tr}(\rho x)$ for all x in \mathcal{A} . More generally, we call a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ (where \mathcal{B} is a von Neumann algebra) normal if whenever x_α increases to x for a net x_α of positive elements from \mathcal{A} , one has that $\Phi(x_\alpha)$ increases to $\Phi(x)$ in \mathcal{B} . It is known that a positive linear map is normal if and only if it is continuous with respect to the ultra-weak-topology. In view of this fact, we shall say that a bounded linear map between two von Neumann algebras is normal if it is continuous with respect to the respective ultra-weak topologies.

1.1.3 Free product and tensor product

If $(\mathcal{A}_i)_{i \in I}$ is a family of unital C^* algebras, then their unital C^* algebra free product $*_{i \in I} \mathcal{A}_i$ is the unique C^* algebra \mathcal{A} together with unital $*$ -homomorphism $\psi_i : \mathcal{A}_i \rightarrow \mathcal{A}$ such that given any unital C^* algebra \mathcal{B} and unital $*$ -homomorphisms $\phi_i : \mathcal{A}_i \rightarrow \mathcal{B}$ there exists a unique unital $*$ -homomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\phi_i = \Phi \circ \psi_i$.

Remark 1.1.6. *It is a direct consequence of the above definition that given a family of C^* homomorphisms ϕ_i from \mathcal{A}_i to \mathcal{B} , there exists a C^* homomorphism $*_i \phi_i$ such that $(*_i \phi_i) \circ \psi_i = \phi_i$ for all i .*

Remark 1.1.7. *We recall that for discrete groups $\{G_i\}_{i \in I}$, $C^*(*_i G_i) \cong *_i C^*(G_i)$.*

For \mathcal{A} and \mathcal{B} two algebras, we will denote the algebraic tensor product of \mathcal{A} and \mathcal{B} by the symbol $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$. When \mathcal{A} and \mathcal{B} are C^* algebras, there is more than one norm on $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$ so that the completion with respect to that norm is a C^* algebra. Throughout this thesis, we will work with the so called injective tensor product, that is, the completion of $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$ with respect to the norm given on $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$ by $\|\sum_{i=1}^n a_i \otimes b_i\| = \sup \|\sum_{i=1}^n \pi_1(a_i) \otimes \pi_2(b_i)\|_{\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$ where a_i is in \mathcal{A} , b_i is in \mathcal{B} and the supremum runs over all possible choices of (π_1, \mathcal{H}_1) , (π_2, \mathcal{H}_2) where $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces and $\pi_1 : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_1)$ and $\pi_2 : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_2)$ are $*$ -homomorphisms. When $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_1)$, $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H}_2)$ are von Neumann algebras, then by the notation $\mathcal{A} \otimes \mathcal{B}$, we mean the von Neumann algebra tensor product, that is, the WOT closure of $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. We refer to [56] for more details.

We now prove a useful general fact.

Lemma 1.1.8. *Let \mathcal{A} be a C^* algebra and ω, ω_j ($j = 1, 2, \dots$) be states on \mathcal{A} such that $\omega_j \rightarrow \omega$ in the weak- $*$ topology of \mathcal{A}^* . Then for any separable Hilbert space \mathcal{H} and for all Y in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$, we have $(\text{id} \otimes \omega_j)(Y) \rightarrow (\text{id} \otimes \omega)(Y)$ in the strong operator topology.*

Proof: Clearly, $(\text{id} \otimes \omega_j)(Y) \rightarrow (\text{id} \otimes \omega)(Y)$ (in the strong operator topology) for all Y in $\text{Fin}(\mathcal{H}) \otimes_{\text{alg}} \mathcal{A}$, where $\text{Fin}(\mathcal{H})$ denotes the set of finite rank operators on \mathcal{H} . Using the strict density of $\text{Fin}(\mathcal{H}) \otimes_{\text{alg}} \mathcal{A}$ in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$, we choose, for a given Y in

$\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$, ξ in \mathcal{H} with $\|\xi\| = 1$, and $\delta > 0$, an element Y_0 in $\text{Fin}(\mathcal{H}) \otimes_{\text{alg}} \mathcal{A}$ such that $\|(Y - Y_0)(|\xi\rangle\langle\xi| \otimes 1)\| < \delta$. Thus,

$$\begin{aligned}
& \|(\text{id} \otimes \omega_j)(Y)\xi - (\text{id} \otimes \omega)(Y)\xi\| \\
&= \|(\text{id} \otimes \omega_j)(Y(|\xi\rangle\langle\xi| \otimes 1))\xi - (\text{id} \otimes \omega)(Y(|\xi\rangle\langle\xi| \otimes 1))\xi\| \\
&\leq \|(\text{id} \otimes \omega_j)(Y_0(|\xi\rangle\langle\xi| \otimes 1))\xi - (\text{id} \otimes \omega)(Y_0(|\xi\rangle\langle\xi| \otimes 1))\xi\| \\
&+ 2\|(Y - Y_0)(|\xi\rangle\langle\xi| \otimes 1)\| \\
&\leq \|(\text{id} \otimes \omega_j)(Y_0(|\xi\rangle\langle\xi| \otimes 1))\xi - (\text{id} \otimes \omega)(Y_0(|\xi\rangle\langle\xi| \otimes 1))\xi\| + 2\delta,
\end{aligned}$$

from which it follows that $(\text{id} \otimes \omega_j)(Y) \rightarrow (\text{id} \otimes \omega)(Y)$ in the strong operator topology. \square

Let \mathcal{A} and \mathcal{B} be two unital $*$ -algebras. Then a linear map T from \mathcal{A} to \mathcal{B} is called n -positive if $T \otimes \text{Id}_n : \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{B} \otimes \mathcal{M}_n(\mathbb{C})$ is positive for all $k \leq n$ but not necessarily for $k = n + 1$. T is said to be completely positive (CP for short) if it is n -positive for all n . It is a well known result that for a CP map $T : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, one has the following operator inequality for all a in \mathcal{A} :

$$T(a)^*T(a) \leq \|T(1)\| T(a^*a). \quad (1.1.1)$$

Tensor product and composition of two CP maps are CP. The following is an useful result about CP maps.

Proposition 1.1.9. *If A and B are C^* algebras with A commutative, ϕ is a positive map from A to B , then ϕ is CP. The same holds if ϕ is from B to A .*

1.1.4 Hilbert modules

Given a $*$ -subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ (where \mathcal{H} is a Hilbert space), a semi-Hilbert \mathcal{A} module E is a right \mathcal{A} -module equipped with a sesquilinear map $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathcal{A}$ satisfying $\langle x, y \rangle^* = \langle y, x \rangle$, $\langle x, ya \rangle = \langle x, y \rangle a$ and $\langle x, x \rangle \geq 0$ for x, y in E and a in \mathcal{A} . A semi Hilbert module is called a pre-Hilbert module if $\langle x, x \rangle = 0$ if and only if $x = 0$. It is called a Hilbert module if furthermore, \mathcal{A} is a C^* algebra and E is complete in the norm $x \mapsto \|\langle x, x \rangle\|^{\frac{1}{2}}$ where $\|\cdot\|$ is the C^* norm of \mathcal{A} .

The simplest examples of Hilbert \mathcal{A} modules are the so called trivial \mathcal{A} modules of the form $\mathcal{H} \otimes \mathcal{A}$ where \mathcal{H} is a Hilbert space with an \mathcal{A} valued sesquilinear form defined on $\mathcal{H} \otimes_{\text{alg}} \mathcal{A}$ by : $\langle \xi \otimes a, \xi' \otimes a' \rangle = \langle \xi, \xi' \rangle a^* a'$. The completion of $\mathcal{H} \otimes_{\text{alg}} \mathcal{A}$ with respect to this pre Hilbert module structure is a Hilbert \mathcal{A} module and is denoted by $\mathcal{H} \otimes \mathcal{A}$.

We recall that for a pre Hilbert \mathcal{A} module E (\mathcal{A} is a C^* algebra), the Cauchy Schwarz inequality holds in the following form: $0 \leq \langle x, y \rangle \langle y, x \rangle \leq \langle x, x \rangle \|\langle y, y \rangle\|$.

Let E and F be two Hilbert \mathcal{A} modules. We say that a \mathbb{C} linear map L from E to F is adjointable if there exists a \mathbb{C} linear map L^* from F to E such that $\langle L(x), y \rangle = \langle x, L^*(y) \rangle$ for all x in E , y in F . We call L^* the adjoint of L . The set of all adjointable maps from E to F is denoted by $\mathcal{L}(E, F)$. In case, $E = F$, we write $\mathcal{L}(E)$ for $\mathcal{L}(E, E)$. For an adjointable map L , both L and L^* are automatically \mathcal{A} -linear and norm bounded maps between Banach spaces. We say that an element L in $\mathcal{L}(E, F)$ is an isometry if $\langle Lx, Ly \rangle = \langle x, y \rangle$ for all x, y in E . L is said to be a unitary if L is isometry and its range is the whole of F . One defines a norm on $\mathcal{L}(E, F)$ by $\|L\| = \sup_{x \in E, \|x\| \leq 1} \|L(x)\|$. $\mathcal{L}(E)$ is a C^* algebra with this norm.

There is a topology on $\mathcal{L}(E, F)$ given by a family of seminorms $\{\|\cdot\|_x, \|\cdot\|_y : x \in E, y \in F\}$ (where $\|t\|_x = \left\| \langle tx, tx \rangle^{\frac{1}{2}} \right\|$ and $\|t\|_y = \left\| \langle t^*y, t^*y \rangle^{\frac{1}{2}} \right\|$) known as the strict topology. For x in E , y in F , we denote by $\theta_{x,y}$ the element of $\mathcal{L}(E, F)$ defined by $\theta_{x,y}(z) = y \langle x, z \rangle$ (where z is in E). The norm closure (in $\mathcal{L}(E, F)$) of the \mathcal{A} linear span of $\{\theta_{x,y} : x \in E, y \in F\}$ is called the set of compact operators and denoted by $\mathcal{K}(E, F)$ and we denote $\mathcal{K}(E, E)$ by $\mathcal{K}(E)$. These are not necessarily compact in the sense of compact operators between two Banach spaces. One has the following important result:

Proposition 1.1.10. *The multiplier algebra $\mathcal{M}(\mathcal{K}(E))$ of $\mathcal{K}(E)$ is isomorphic with $\mathcal{L}(E)$ for any Hilbert module E .*

Using this, for a possibly non-unital C^* algebra \mathcal{B} , we often identify an element V of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{B})$ with the map from \mathcal{H} to $\mathcal{H} \otimes \mathcal{B}$ which sends a vector ξ of \mathcal{H} to $V(\xi \otimes 1_{\mathcal{B}})$ in $\mathcal{H} \otimes \mathcal{B}$.

Given a Hilbert space \mathcal{H} and a C^* algebra \mathcal{A} , and a unitary element U of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$, we shall denote by $\alpha_U (\equiv ad_U)$ the $*$ -homomorphism $\alpha_U(X) = \tilde{U}(X \otimes 1)\tilde{U}^*$ for X belonging to $\mathcal{B}(\mathcal{H})$. For a not necessarily bounded, densely defined (in the weak operator topology) linear functional τ on $\mathcal{B}(\mathcal{H})$, we say that α_U preserves τ if α_U maps a suitable (weakly) dense $*$ -subalgebra (say \mathcal{D}) in the domain of τ into $\mathcal{D} \otimes_{\text{alg}} \mathcal{A}$ and $(\tau \otimes \text{id})(\alpha_U(x)) = \tau(x) \cdot 1_{\mathcal{A}}$ for all x in \mathcal{D} . When τ is bounded and normal, this is equivalent to $(\tau \otimes \text{id})(\alpha_U(x)) = \tau(x) 1_{\mathcal{A}}$ for all x belonging to $\mathcal{B}(\mathcal{H})$.

We say that a (possibly unbounded) operator T on \mathcal{H} commutes with U if $T \otimes I$ (with the natural domain) commutes with \tilde{U} . Sometimes such an operator will be called U -equivariant.

1.2 Quantum Groups

In this section, we will recall the basics of Hopf algebras and then define compact quantum groups (as in [67], [66]). After that, we will discuss a few examples of

quantum groups and the concept of an action of a compact quantum group on a C^* algebra. For more detailed discussion, we refer to [37], [35], [15] [42] and references therein. In this thesis, we will be concerned about compact quantum groups only. For other types of quantum groups, we refer to [37], [38], [58] etc.

1.2.1 Hopf algebras

We recall that an associative algebra with an unit is a vector space A over \mathbb{C} together with two linear maps $m : A \otimes A \rightarrow A$ called the multiplication or the product and $\eta : \mathbb{C} \rightarrow A$ called the unit such that $m(m \otimes \text{id}) = m(\text{id} \otimes m)$ and $m(\eta \otimes \text{id}) = \text{id} = m(\text{id} \otimes \eta)$. Dualizing this, we get the following definition.

Definition 1.2.1. A **coalgebra** A is a vector space over \mathbb{C} equipped with two linear maps $\Delta : A \rightarrow A \otimes A$ called the comultiplication or coproduct and $\epsilon : A \rightarrow \mathbb{C}$ such that

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta,$$

$$(\epsilon \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \epsilon)\Delta.$$

Definition 1.2.2. Let $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ be co algebras. A \mathbb{C} linear mapping $\phi : A \rightarrow B$ is said to be a **cohomomorphism** if

$$\Delta_B \circ \phi = (\phi \otimes \phi)\Delta_A$$

$$\epsilon_A = \epsilon_B \circ \phi$$

Let σ denote the flip map : $A \otimes A \rightarrow A \otimes A$ given by $\sigma(a \otimes b) = b \otimes a$.

Definition 1.2.3. A coalgebra is said to be **cocommutative** if $\sigma \circ \Delta = \Delta$.

Definition 1.2.4. A linear subspace B of A is a **subcoalgebra** of A if $\Delta(B) \subseteq B \otimes B$.

Definition 1.2.5. A \mathbb{C} linear subspace \mathcal{I} of A is called a **coideal** if $\Delta(\mathcal{I}) \subseteq A \otimes \mathcal{I} + \mathcal{I} \otimes A$ and $\epsilon(\mathcal{I}) = \{0\}$.

If \mathcal{I} is a coideal of A , the quotient vector space A/\mathcal{I} becomes a coalgebra with comultiplication and counit induced from A .

Sweedler notation

We introduce the so called Sweedler notation for comultiplication. If a is an element of a coalgebra \mathcal{A} , the element $\Delta(a)$ in $\mathcal{A} \otimes \mathcal{A}$ is a finite sum $\Delta(a) = \sum_i a_{1i} \otimes a_{2i}$ where a_{1i}, a_{2i} belongs to \mathcal{A} . Moreover, the representation of $\Delta(a)$ is not unique. For notational simplicity we shall suppress the index i and write the above sum symbolically

as $\Delta(a) = a_{(1)} \otimes a_{(2)}$. Here the subscripts (1) and (2) refer to the corresponding tensor factors.

Definition 1.2.6. A vector space \mathcal{A} is called a bialgebra if it is an algebra and a coalgebra along with the condition that $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ and $\epsilon : \mathcal{A} \rightarrow \mathbb{C}$ are algebra homomorphisms (equivalently, $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ and $\eta : \mathbb{C} \rightarrow \mathcal{A}$ are coalgebra co-homomorphisms).

Definition 1.2.7. A bialgebra \mathcal{A} is called a Hopf algebra if there exists a linear map $\kappa : \mathcal{A} \rightarrow \mathcal{A}$ called the antipode or the coinverse of \mathcal{A} , such that $m \circ (\kappa \otimes \text{id})\Delta = \eta \circ \epsilon = m \circ (\text{id} \otimes \kappa) \circ \Delta$.

Dual Hopf algebra

Let us consider a finite dimensional Hopf algebra \mathcal{A} . Then the dual vector space \mathcal{A}' is an algebra with respect to the multiplication $fg(a) = (f \otimes g)\Delta(a)$. Moreover, for f in \mathcal{A}' , one defines the functional $\Delta(f) \in (\mathcal{A} \otimes \mathcal{A})'$ by $\Delta(f)(a \otimes b) = f(ab)$, a, b in \mathcal{A} . Since \mathcal{A} is finite dimensional, $(\mathcal{A} \otimes \mathcal{A})' = \mathcal{A}' \otimes \mathcal{A}'$ and so $\Delta(f)$ belongs to $\mathcal{A}' \otimes \mathcal{A}'$. Then the algebra \mathcal{A}' equipped with the comultiplication Δ , antipode κ defined by $(\kappa f)(a) = f(\kappa(a))$, counit $\epsilon_{\mathcal{A}'}$ defined by $\epsilon_{\mathcal{A}'}(f) = f(1)$ and $1_{\mathcal{A}'}(a) = \epsilon(a)$ gives a Hopf algebra. This is called the dual Hopf algebra of \mathcal{A} .

Definition 1.2.8. A Hopf $*$ -algebra is a Hopf algebra $(\mathcal{A}, \Delta, \kappa, \epsilon)$ endowed with an involution $*$ which maps a to an element denoted by a^* satisfying the following properties:

1. For all a, b in \mathcal{A} , α, β in \mathbb{C} , $(\alpha a + \beta b)^* = \bar{\alpha}a^* + \bar{\beta}b^*$, $(a^*)^* = a$, $(a.b)^* = b^*a^*$.
2. $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is a $*$ -homomorphism which means that $\Delta(a^*) = \Delta(a)^*$ where the involution on $\mathcal{A} \otimes \mathcal{A}$ is defined by $(a \otimes b)^* = a^* \otimes b^*$.

Proposition 1.2.9. In any Hopf $*$ -algebra $(\mathcal{A}, \Delta, \kappa, \epsilon)$, we have

1. $\epsilon(a^*) = \overline{\epsilon(a)}$ for all a in \mathcal{A} .
2. $\kappa(\kappa(a^*)^*) = a$ for all a in \mathcal{A} .

We recall that the dual algebra \mathcal{A}' of a Hopf $*$ -algebra \mathcal{A} is a $*$ -algebra with involution defined by

$$f^*(a) = \overline{f(\kappa(a)^*)}, \text{ for } f \text{ in } \mathcal{A}'.$$

Dual Pairing

A left action of a Hopf $*$ -algebra $(\mathcal{U}, \Delta_{\mathcal{U}}, \kappa_{\mathcal{U}}, \epsilon_{\mathcal{U}})$ on another Hopf $*$ -algebra $(\mathcal{A}, \Delta_{\mathcal{A}}, \kappa_{\mathcal{A}}, \epsilon_{\mathcal{A}})$ is a bilinear form $\triangleright : \mathcal{U} \times \mathcal{A} \rightarrow \mathbb{C}$ if the following conditions hold:

$$(1)f \triangleright (a_1 a_2) = \Delta_{\mathcal{U}}(f) \triangleright (a_1 \otimes a_2), \quad (f_1 f_2) \triangleright a = (f_1 \otimes f_2) \triangleright \Delta_{\mathcal{A}}(a);$$

$$(2) f \triangleright 1_{\mathcal{A}} = \epsilon_{\mathcal{U}}(f), \quad 1_{\mathcal{U}} \triangleright a = \epsilon_{\mathcal{A}}(a);$$

$$(3) f^* \triangleright a = \overline{f \triangleright \kappa_{\mathcal{A}}(a)^*} \text{ (equivalently } f \triangleright a^* = \overline{\kappa_{\mathcal{U}}(f)^* \triangleright a})$$

for all f, f_1, f_2 in \mathcal{U} and a, a_1, a_2 in \mathcal{A} .

Similarly, a right action of a Hopf $*$ -algebra $(\mathcal{U}, \Delta_{\mathcal{U}}, \kappa_{\mathcal{U}}, \epsilon_{\mathcal{U}})$ on another Hopf $*$ -algebra $(\mathcal{A}, \Delta_{\mathcal{A}}, \kappa_{\mathcal{A}}, \epsilon_{\mathcal{A}})$ is a bilinear form $\triangleleft : \mathcal{A} \times \mathcal{U} \rightarrow \mathbb{C}$ if the following conditions hold: $a_1 a_2 \triangleleft f = (a_1 \triangleleft f_{(1)})(a_2 \triangleleft f_{(2)})$, $a \triangleleft (f_1 f_2) = \Delta_{\mathcal{A}}(a) \triangleleft (f_1 \otimes f_2)$, $1_{\mathcal{A}} \triangleleft f = \epsilon_{\mathcal{U}}(f)$, $a \triangleleft 1_{\mathcal{U}} = \epsilon_{\mathcal{A}}(a)$, $a \triangleleft f^* = \overline{\kappa_{\mathcal{A}}(a)^* \triangleleft f}$ (equivalently $a^* \triangleleft f = \overline{a \triangleleft \kappa_{\mathcal{U}}(f)^*}$) for all f, f_1, f_2 in \mathcal{U} and a, a_1, a_2 in \mathcal{A} .

$\mathcal{U} = \mathcal{A}'$ gives a particular case of this duality pairing.

1.2.2 Compact Quantum Groups: basic definitions and examples

Definition 1.2.10. A compact quantum group (to be abbreviated as CQG from now on) is given by a pair (\mathcal{S}, Δ) , where \mathcal{S} is a unital separable C^* algebra equipped with a unital C^* -homomorphism $\Delta : \mathcal{S} \rightarrow \mathcal{S} \otimes \mathcal{S}$ (where \otimes denotes the injective tensor product) satisfying

- (ai) $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ (co-associativity), and
- (aii) each of the linear spans of $\Delta(\mathcal{S})(\mathcal{S} \otimes 1)$ and of $\Delta(\mathcal{S})(1 \otimes \mathcal{S})$ are norm-dense in $\mathcal{S} \otimes \mathcal{S}$.

It is well-known (see [67], [66]) that there is a canonical dense $*$ -subalgebra \mathcal{S}_0 of \mathcal{S} , consisting of the matrix elements of the finite dimensional unitary (co)-representations (to be defined shortly) of \mathcal{S} , and maps $\epsilon : \mathcal{S}_0 \rightarrow \mathbb{C}$ (co-unit) and $\kappa : \mathcal{S}_0 \rightarrow \mathcal{S}_0$ (antipode) defined on \mathcal{S}_0 which make \mathcal{S}_0 a Hopf $*$ -algebra.

The following theorem is the analogue of Gelfand Naimark duality for commutative CQG s.

Proposition 1.2.11. Let G be a compact group. Let $C(G)$ be the algebra of continuous functions on G . If we define Δ by $\Delta(f)(g, h) = f(g.h)$ for f in $C(G)$, g, h in G , then this defines a CQG structure on $C(G)$.

Conversely, let (\mathcal{S}, Δ) be a commutative CQG. Let $H(\mathcal{S})$ denote the Gelfand spectrum of \mathcal{S} and endow it with the product structure given by $\chi \chi' = (\chi \otimes \chi') \Delta$ where χ, χ' are in $H(\mathcal{S})$. Then $H(\mathcal{S})$ is a compact group.

Remark 1.2.12. In [57], A Van Daele removed Woronowicz's separability assumption (in [67]) for the C^* algebra of the underlying compact quantum group. We remark

that although we assume that CQG s are separable, most of the results in this thesis go through in the non separable case also.

Definition 1.2.13. Let $(\mathcal{S}, \Delta_{\mathcal{S}})$ be a compact quantum group. A vector space M is said to be an algebraic \mathcal{S} co-module (or \mathcal{S} co-module) if there exists a linear map $\alpha : M \rightarrow M \otimes_{\text{alg}} \mathcal{S}_0$ such that

1. $\alpha \otimes \text{id})\alpha = (\text{id} \otimes \Delta_{\mathcal{S}})\alpha$,
2. $(\text{id} \otimes \epsilon)\alpha(m) = m$ for all m in M .

In the notations as above, let us define $\tilde{\alpha} : M \otimes \mathcal{S} \rightarrow M \otimes \mathcal{S}$ by $\tilde{\alpha} = (\text{id} \otimes m)(\alpha \otimes \text{id})$. Then we claim that $\tilde{\alpha}$ is invertible with the inverse given by $T(m \otimes q) = (\text{id} \otimes \kappa)\alpha(m)(1 \otimes q)$, where m is in M , q is in \mathcal{S} . As T is defined to be \mathcal{S}_0 linear, it is enough to check that $\tilde{\alpha}T(m \otimes 1) = m \otimes 1$.

$$\begin{aligned}
 \tilde{\alpha}T(m \otimes 1) &= \tilde{\alpha}(m_{(1)} \otimes \kappa(m_{(2)})1) \\
 &= m_{(1)(1)} \otimes m_{(1)(2)}\kappa(m_{(2)}) \\
 &= (\text{id} \otimes m(\text{id} \otimes \kappa)\Delta)\alpha(m) \\
 &= (\text{id} \otimes \epsilon(.).1)\alpha(m) \\
 &= m \otimes 1.
 \end{aligned}$$

Similarly, $T\tilde{\alpha} = \text{id}$. Thus,

$$T = \tilde{\alpha}^{-1}. \quad (1.2.1)$$

Definition 1.2.14. A morphism from a CQG $(\mathcal{S}_1, \Delta_1)$ to another CQG $(\mathcal{S}_2, \Delta_2)$ is a unital C^* homomorphism $\pi : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ such that

$$(\pi \otimes \pi)\Delta_1 = \Delta_2\pi.$$

It follows that in such a case, π preserves the Hopf $*$ -algebra structures, that is, we have

$$\pi((\mathcal{S}_1)_0) \subseteq (\mathcal{S}_2)_0, \quad \pi\kappa_1 = \kappa_2\pi, \quad \epsilon_2\pi = \epsilon_1,$$

where κ_1, ϵ_1 denotes the antipode and counit of \mathcal{S}_1 respectively while κ_2, ϵ_2 denotes those of \mathcal{S}_2 .

Definition 1.2.15. A Woronowicz C^* -subalgebra of a CQG (\mathcal{S}_1, Δ) is a C^* subalgebra \mathcal{S}_2 of \mathcal{S}_1 such that $(\mathcal{S}_2, \Delta|_{\mathcal{S}_2})$ is a CQG such that the inclusion map from $\mathcal{S}_2 \rightarrow \mathcal{S}_1$ is a morphism of CQG s.

Definition 1.2.16. A Woronowicz C^* -ideal of a CQG (\mathcal{S}, Δ) is a C^* ideal J of \mathcal{S} such that $\Delta(J) \subseteq \text{Ker}(\pi \otimes \pi)$, where π is the quotient map from \mathcal{S} to \mathcal{S}/J .

It can be easily seen that a kernel of a CQG morphism is a Woronowicz C^* -ideal.

We recall the following isomorphism theorem.

Proposition 1.2.17. *The quotient of a CQG (S, Δ) by a Woronowicz C^* -ideal \mathcal{I} has a unique CQG structure such that the quotient map π is a morphism of CQG s. More precisely, the coproduct $\tilde{\Delta}$ on S/\mathcal{I} is given by $\tilde{\Delta}(s + \mathcal{I}) = (\pi \otimes \pi)\Delta(s)$.*

Definition 1.2.18. *A CQG (S', Δ') is called a quantum subgroup of another CQG (S, Δ) if there is a Woronowicz C^* -ideal J of S such that $(S', \Delta') \cong (S, \Delta)/J$.*

Let us mention a convention which we are going to follow. We shall use most of the terminologies of [59], for example Woronowicz C^* -subalgebra, Woronowicz C^* -ideal etc, however with the exception that we shall call the Woronowicz C^* algebras just compact quantum groups, and not use the term compact quantum groups for the dual objects as done in [59].

Let (S, Δ) be a compact quantum group. Then there exists a state h on S , to be called a **Haar state** on S such that $(h \otimes \text{id})\Delta(s) = (\text{id} \otimes h)\Delta(s) = h(s).1$. We recall that unlike the group case, h may not be faithful. But on the dense Hopf $*$ -algebra S_0 mentioned above, it is faithful. We have the following result.

Proposition 1.2.19. *Let $i : S_1 \rightarrow S_2$ be an injective morphism of CQG s. Then the Haar state on S_1 is the restriction of that of S_2 on S_1 .*

Remark 1.2.20. *In general, the Haar state might not be tracial. In fact, there exists a multiplicative linear functional denoted by f_1 in [66] such that $h(ab) = h(b(f_1 \triangleleft a \triangleright f_1))$. Moreover, from Theorem 1.5 of [67], it follows that the Haar state of a CQG is tracial if and only if $\kappa^2 = \text{id}$.*

Co-Representations of a compact quantum group

Definition 1.2.21. *A co-representation of a compact quantum group (S, Δ) on a Hilbert space \mathcal{H} is a map U from \mathcal{H} to the Hilbert S module $\mathcal{H} \otimes S$ such that the element \tilde{U} belonging to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes S)$ given by $\tilde{U}(\xi \otimes b) = U(\xi)(1 \otimes b)$ (ξ in \mathcal{H} , b in S) satisfies*

$$(\text{id} \otimes \Delta)\tilde{U} = \tilde{U}_{(12)}\tilde{U}_{(13)},$$

where for an operator X in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ we have denoted by $X_{(12)}$ and $X_{(13)}$ the operators $X \otimes I_{\mathcal{H}_2}$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2)$, and $\Sigma_{23}X_{(12)}\Sigma_{23}$ respectively and Σ_{23} is the unitary on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$ which flips the two copies of \mathcal{H}_2 .

If \tilde{U} is an unitary element of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes S)$, then U is called a unitary co-representation.

From now on, we will drop the term co in the word co-representation unless there is any confusion.

Remark 1.2.22. *Let π be a CQG morphism from a CQG $(\mathcal{S}_1, \Delta_1)$ to another CQG $(\mathcal{S}_2, \Delta_2)$. Then for every unitary representation U of \mathcal{S}_1 , $(\text{id} \otimes \pi)U$ is a unitary representation of \mathcal{S}_2 .*

Following the definitions given in the last part of subsection 1.1.4 and a unitary representation U of a CQG on a Hilbert space \mathcal{H} , and a not necessarily bounded, densely defined (in the weak operator topology) linear functional τ on $\mathcal{B}(\mathcal{H})$, we will use the notation α_U and the terms “ α_U preserves τ ” and “ U equivariant ” throughout this thesis.

A CQG (\mathcal{S}, Δ) has a distinguished representation which corresponds to the right regular representation in the group case. Let \mathcal{H} be the GNS space of \mathcal{S} associated with the Haar state h , ξ_0 be the associated cyclic vector and \mathcal{K} be a Hilbert space on which \mathcal{S} acts faithfully and non-degenerately. There is a unitary operator u on $\mathcal{H} \otimes \mathcal{K}$ defined by $u(a\xi_0 \otimes \eta) = \Delta(a)(\xi_0 \otimes \eta)$ when a is in \mathcal{S} , η is in \mathcal{K} . Then u can be shown to be an element of multiplier of $\mathcal{K}(\mathcal{H}) \otimes \mathcal{S}$ and called the right regular representation of \mathcal{S} .

Let v be a representation of a CQG (\mathcal{S}, Δ) on a Hilbert space \mathcal{H} . A closed subspace \mathcal{H}_1 of \mathcal{H} is said to be invariant if $(e \otimes 1)v(e \otimes 1) = v(e \otimes 1)$, where e is the orthogonal projection onto this subspace. The representation v is called irreducible if the only invariant subspaces are $\{0\}$ and \mathcal{H} . It is clear that one can make sense of direct sum of (co)-representations in this case also. Moreover, for two representations v and w of a CQG (\mathcal{S}, Δ) on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , the tensor product of v and w is given by the element $v_{(13)}w_{(23)}$. The intertwiner between v and w is an element x in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $(x \otimes 1)v = w(x \otimes 1)$. The set of intertwiners between v and w is denoted by $\text{Mor}(v, w)$. Two representations are said to be equivalent if there is an invertible intertwiner. They are unitarily equivalent if the intertwiner can be chosen to be unitary.

Just like the case of compact groups, CQG s have an analogous Peter Weyl theory which corresponds to the usual Peter Weyl theory in the group case. We will give a sketch of it by mentioning the main results and refer to [41], [66] and [67] for the details.

Let v be a unitary representation of (\mathcal{S}, Δ) on \mathcal{H} . If \mathcal{H}_1 is an invariant subspace, then the orthogonal complement of \mathcal{H}_1 is also invariant. Any non degenerate finite dimensional representation is equivalent with a unitary representation.

Every irreducible unitary representation of a CQG is contained in the regular representation. Let v be a representation on a finite dimensional Hilbert space \mathcal{H} . If we denote the matrix units in $\mathcal{B}(\mathcal{H})$ by (e_{pq}) , we can write $v = \sum e_{pq} \otimes v_{pq}$. v_{pq} are called the matrix elements of the finite dimensional representation v . Define $\bar{v} = \sum e_{pq} \otimes v_{pq}^*$. Then \bar{v} is a representation and is called the adjoint of v . It can be shown that if v is a finite dimensional irreducible representation, then \bar{v} is also irreducible. Moreover, for

an irreducible unitary representation, its adjoint is equivalent with a unitary representation.

The subspace spanned by the matrix elements of finite dimensional unitary representations is denoted by \mathcal{S}_0 . Firstly, \mathcal{S}_0 is a subalgebra as the product of two matrix elements of finite dimensional unitary representations is a matrix element of the tensor product of these representations. Moreover, as the adjoint of a finite dimensional unitary representation is equivalent with a unitary representation, \mathcal{S}_0 is $*$ invariant. We note that 1 is in \mathcal{S}_0 as 1 is a representation. Now, we will recall some basic facts about the subalgebra \mathcal{S}_0 . We will denote the Haar state of \mathcal{S} by h .

Proposition 1.2.23. (1) \mathcal{S}_0 is a dense $*$ -subalgebra of \mathcal{S} .

(2) Let $\{u^\alpha : \alpha \in I\}$ be a complete set of mutually inequivalent, irreducible unitary representations. We will denote the representation space and dimension of u^α by \mathcal{H}_α and $n(\alpha)$ respectively. Then the Schur's orthogonality relation takes the following form:

For any α in I , there is a positive invertible operator F^α acting on \mathcal{H}_α such that for any α, β in I and $1 \leq j, q \leq n(\alpha)$, $1 \leq i, p \leq n(\beta)$

$$h((u_{ip}^\beta)^* u_{jq}^\alpha) = \delta_{\alpha\beta} \delta_{pq} F_{ij}^\alpha.$$

(3) $\{u_{pq}^\alpha : \alpha \in I, 1 \leq p, q \leq n(\alpha)\}$ form a basis for \mathcal{S}_0 .

(4) Moreover, Δ maps \mathcal{S}_0 into $\mathcal{S}_0 \otimes \mathcal{S}_0$. In fact, Δ is given by $\Delta(u_{pq}^\alpha) = \sum_{k=1}^{n_\alpha} u_{pk}^\alpha \otimes u_{kq}^\alpha$. A counit and an antipode are defined on \mathcal{S}_0 respectively by the formulae,

$$\epsilon(u_{pq}^\alpha) = \delta_{pq}, \quad \kappa(u_{pq}^\alpha) = (u_{qp}^\alpha)^*.$$

It follows that \mathcal{S}_0 becomes a Hopf $*$ -algebra.

A **compact matrix quantum group** is a CQG such that there exists a distinguished unitary irreducible representation called the fundamental representation such that the $*$ -algebra spanned by its matrix elements is a dense Hopf $*$ -subalgebra of the CQG.

We now discuss the **free product of CQG** s which were developed in [59]. Let $(\mathcal{S}_1, \Delta_1)$ and $(\mathcal{S}_2, \Delta_2)$ be two CQG s. Let i_1 and i_2 denote the canonical injections of \mathcal{S}_1 and \mathcal{S}_2 into the C^* algebra $\mathcal{S}_1 * \mathcal{S}_2$. Put $\rho_1 = (i_1 \otimes i_1)\Delta_1$ and $\rho_2 = (i_2 \otimes i_2)\Delta_2$. By the universal property of $\mathcal{S}_1 * \mathcal{S}_2$, there exists a map $\Delta : \mathcal{S}_1 * \mathcal{S}_2 \rightarrow (\mathcal{S}_1 * \mathcal{S}_2) \otimes (\mathcal{S}_1 * \mathcal{S}_2)$ such that $\Delta i_1 = \rho_1$ and $\Delta i_2 = \rho_2$. It can be shown that Δ indeed has the required properties so that (\mathcal{S}, Δ) is a CQG.

Let $\{\mathcal{S}_n\}_{n \in \mathbb{N}}$ be an inductive sequence of CQG s, where the connecting morphisms π_{mn} from \mathcal{S}_n to \mathcal{S}_m ($n < m$) are injective morphisms of CQG s. Then from Proposition

3.1 of [59], we have that the inductive limit \mathcal{S}_0 of \mathcal{S}_n s has a unique CQG structure with the following property: for any CQG \mathcal{S}' and any family of CQG morphisms $\phi_n : \mathcal{S}_n \rightarrow \mathcal{S}'$ such that $\phi_m \pi_{mn} = \phi_n$, the uniquely defined morphism $\lim_n \phi_n$ in the category of unital C^* algebras is a morphism in the category of CQG s.

Combining the above two results, it follows that the free product C^* algebra of an arbitrary sequence of CQG s has a natural CQG structure.

Moreover, the following result was derived in [59].

Proposition 1.2.24. *Let Γ_1, Γ_2 be a discrete abelian groups. Then the natural isomorphisms $C^*(\Gamma_1) \cong C(\widehat{\Gamma_1})$ and $C^*(\Gamma_1) * C^*(\Gamma_2) \cong C^*(\Gamma_1 * \Gamma_2)$ are isomorphism of CQG s.*

Let i_1 and i_2 be the inclusion of CQG s \mathcal{S}_1 and \mathcal{S}_2 into $\mathcal{S}_1 * \mathcal{S}_2$. If U_1 and U_2 are unitary representations of CQG s \mathcal{S}_1 and \mathcal{S}_2 on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively, then the **free product representation** of U_1 and U_2 is a representation of the CQG $\mathcal{S}_1 * \mathcal{S}_2$ on the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ given by the $\mathcal{S}_1 * \mathcal{S}_2$ valued matrix

$$\begin{pmatrix} (\text{id} \otimes i_1)U_1 & 0 \\ 0 & (\text{id} \otimes i_2)U_2 \end{pmatrix}.$$

Similarly, the free product representation of an arbitrary sequence of CQG representations are defined.

The inductive limit of an arbitrary sequence of CQG s has the structure of a CQG. The following lemma is probably known, but we include the proof (taken from [9]) for the sake of completeness.

Lemma 1.2.25. *Suppose that $(\mathcal{S}_n)_{n \in \mathbb{N}}$ is a sequence of CQG s and for each n, m in \mathbb{N} , $n \leq m$ there is a CQG morphism $\pi_{n,m} : \mathcal{S}_n \rightarrow \mathcal{S}_m$ with the compatibility property*

$$\pi_{m,k} \circ \pi_{n,m} = \pi_{n,k}, \quad n \leq m \leq k.$$

Then the inductive limit of C^ -algebras $(\mathcal{S}_n)_{n \in \mathbb{N}}$ has a canonical structure of a CQG. It will be denoted \mathcal{S}_∞ or $\lim_{n \in \mathbb{N}} \mathcal{S}_n$. It has the following universality property: for any CQG (\mathcal{S}, Δ) such that there are CQG morphisms $\pi_n : \mathcal{S}_n \rightarrow \mathcal{S}$ satisfying for all $m, n \in \mathbb{N}$, $m \geq n$ the equality $\pi_m \circ \pi_{n,m} = \pi_n$, there exists a unique CQG morphism $\pi_\infty : \mathcal{S}_\infty \rightarrow \mathcal{S}$ such that $\pi_n = \pi_\infty \circ \pi_{n,\infty}$ for all $n \in \mathbb{N}$, where we have denoted by $\pi_{n,\infty}$ the canonical unital C^* -homomorphism from \mathcal{S}_n into \mathcal{S}_∞ .*

Proof:

Let us denote the coproduct on \mathcal{S}_n by Δ_n . We consider the unital C^* -homomorphism $\rho_n : \mathcal{S}_n \rightarrow \mathcal{S}_\infty \otimes \mathcal{S}_\infty$ given by $\rho_n = (\pi_{n,\infty} \otimes \pi_{n,\infty}) \circ \Delta_n$, and observe that these maps do

satisfy the compatibility property:

$$\rho_m \circ \pi_{n,m} = \rho_n \quad \forall n \leq m.$$

Thus, by the general properties of the C^* -algebraic inductive limit, we have a unique unital C^* -homomorphism $\Delta_\infty : \mathcal{S}_\infty \rightarrow \mathcal{S}_\infty \otimes \mathcal{S}_\infty$ satisfying $\Delta_\infty \circ \pi_{n,\infty} = \rho_n$ for all n . We claim that $(\mathcal{S}_\infty, \Delta_\infty)$ is a CQG.

We first check that Δ_∞ is coassociative. It is enough to verify the coassociativity on the dense set $\cup_n \pi_{n,\infty}(\mathcal{S}_n)$. Indeed, for $s = \pi_{n,\infty}(a)$ ($a \in \mathcal{S}_n$), by using $\Delta_\infty \circ \pi_{n,\infty} = (\pi_{n,\infty} \otimes \pi_{n,\infty}) \circ \Delta_n$, we have the following:

$$\begin{aligned} & (\Delta_\infty \otimes \text{id})\Delta_\infty(\pi_{n,\infty}(a)) \\ &= (\Delta_\infty \otimes \text{id})(\pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n(a)) \\ &= (\pi_{n,\infty} \otimes \pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n \otimes \text{id})(\Delta_n(a)) \\ &= (\pi_{n,\infty} \otimes \pi_{n,\infty} \otimes \pi_{n,\infty})(\text{id} \otimes \Delta_n)(\Delta_n(a)) \\ &= (\pi_{n,\infty} \otimes (\pi_{n,\infty} \otimes \pi_{n,\infty}) \circ \Delta_n)(\Delta_n(a)) \\ &= (\pi_{n,\infty} \otimes \Delta_\infty \circ \pi_{n,\infty})(\Delta_n(a)) \\ &= (\text{id} \otimes \Delta_\infty)((\pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n(a))) \\ &= (\text{id} \otimes \Delta_\infty)(\Delta_\infty(\pi_{n,\infty}(a))) \end{aligned}$$

which proves the coassociativity.

Finally, we need to verify the quantum cancellation properties. Note that to show that $\Delta_\infty(\mathcal{S}_\infty)(1 \otimes \mathcal{S}_\infty)$ is dense in $\mathcal{S}_\infty \otimes \mathcal{S}_\infty$ it is enough to show that the above assertion is true with \mathcal{S}_∞ replaced by a dense subalgebra $\bigcup_n \pi_{n,\infty}(\mathcal{S}_n)$.

Using the density of $\Delta_n(\mathcal{S}_n)(1 \otimes \mathcal{S}_n)$ in $\mathcal{S}_n \otimes \mathcal{S}_n$ and the contractivity of the map $\pi_{n,\infty}$ we note that $(\pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n(\mathcal{S}_n)(1 \otimes \mathcal{S}_n))$ is dense in $(\pi_{n,\infty} \otimes \pi_{n,\infty})(\mathcal{S}_n \otimes \mathcal{S}_n)$. This implies that $(\pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n(\mathcal{S}_n))(1 \otimes \pi_{n,\infty}(\mathcal{S}_n))$ is dense in $\pi_{n,\infty}(\mathcal{S}_n) \otimes \pi_{n,\infty}(\mathcal{S}_n)$ and hence $\Delta_\infty(\pi_{n,\infty}(\mathcal{S}_n))(1 \otimes \pi_{n,\infty}(\mathcal{S}_n))$ is dense in $\pi_{n,\infty}(\mathcal{S}_n) \otimes \pi_{n,\infty}(\mathcal{S}_n)$. The proof of the claim now follows by noting that $\pi_{n,\infty}(\mathcal{S}_n) = \pi_{m,\infty}\pi_{n,m}(\mathcal{S}_n) \subseteq \pi_{m,\infty}(\mathcal{S}_m)$ for any $m \geq n$, along with the above observations. The right quantum cancellation property can be shown in the same way.

The proof of the universality property is routine and hence omitted.

□

We note that the proof remains valid for any other indexing set for the net, not necessarily \mathbb{N} .

1.2.3 The CQG $U_\mu(2)$

We now introduce the compact quantum group $U_\mu(2)$. We refer to [37] for more details.

As a unital C^* algebra, $U_\mu(2)$ is generated by 4 elements $u_{11}, u_{12}, u_{21}, u_{22}$ satisfying:

$$u_{11}u_{12} = \mu u_{12}u_{11} \quad (1.2.2)$$

$$u_{11}u_{21} = \mu u_{21}u_{11} \quad (1.2.3)$$

$$u_{12}u_{22} = \mu u_{22}u_{12} \quad (1.2.4)$$

$$u_{21}u_{22} = \mu u_{22}u_{21} \quad (1.2.5)$$

$$u_{12}u_{21} = u_{21}u_{12} \quad (1.2.6)$$

$$u_{11}u_{22} - u_{22}u_{11} = (\mu - \mu^{-1})u_{12}u_{21} \quad (1.2.7)$$

and the condition that the matrix $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ is a unitary. Thus, the above matrix u is the fundamental unitary for $U_\mu(2)$.

The CQG structure is given by

$$\Delta(u_{ij}) = \sum_{k=1,2} u_{ik} \otimes u_{kj}, \quad \kappa(u_{ij}) = u_{ji}^*, \quad \epsilon(u_{ij}) = \delta_{ij}. \quad (1.2.8)$$

The quantum determinant D_μ is defined by

$$D_\mu = u_{11}u_{22} - \mu u_{12}u_{21} = u_{22}u_{11} - \mu^{-1}u_{12}u_{21}. \quad (1.2.9)$$

Then, $D_\mu^* D_\mu = D_\mu D_\mu^* = 1$. Moreover, D_μ belongs to the centre of $U_\mu(2)$.

We mention the following result for future use.

Proposition 1.2.26.

$$\kappa(u_{11}) = u_{22}D_\mu^{-1}, \quad \kappa(u_{12}) = -\mu^{-1}u_{12}D_\mu^{-1}, \quad \kappa(u_{21}) = -\mu u_{21}D_\mu^{-1}, \quad \kappa(u_{22}) = u_{11}D_\mu^{-1}.$$

Proof : By [37] (Proposition 10, Page 314), we have that $\kappa(u_{ij}) = \widetilde{u}_{ij}D_\mu^{-1}$ where \widetilde{u}_{ij} is the (i, j) th entry of a matrix \widetilde{u} satisfying $\widetilde{u}u = u\widetilde{u} = D_\mu I_2$. One can easily check that the matrix $\begin{pmatrix} u_{22} & -\mu^{-1}u_{12} \\ -\mu u_{21} & u_{11} \end{pmatrix}$ satisfies this equation and this proves the Proposition. \square

1.2.4 The CQG $SU_\mu(2)$

Let μ belongs to $[-1, 1]$. The C^* algebra $SU_\mu(2)$ is defined as the universal unital C^* algebra generated by α, γ satisfying:

$$\alpha^* \alpha + \gamma^* \gamma = 1, \quad (1.2.10)$$

$$\alpha \alpha^* + \mu^2 \gamma \gamma^* = 1, \quad (1.2.11)$$

$$\gamma \gamma^* = \gamma^* \gamma, \quad (1.2.12)$$

$$\mu \gamma \alpha = \alpha \gamma, \quad (1.2.13)$$

$$\mu \gamma^* \alpha = \alpha \gamma^*. \quad (1.2.14)$$

The fundamental representation of $SU_\mu(2)$ is given by : $\begin{pmatrix} \alpha & -\mu \gamma^* \\ \gamma & \alpha^* \end{pmatrix}$.

There is a coproduct Δ of $SU_\mu(2)$ given by :

$$\Delta(\alpha) = \alpha \otimes \alpha - \mu \gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

which makes it into a CQG. Let h denote the Haar state and $\mathcal{H} = L^2(SU_\mu(2))$ be the corresponding G.N.S space.

Haar state on $SU_\mu(2)$

We restate the content of Theorem 14, Chapter 4 (page 113) of [37] in a convenient form below. For all $m \geq 1, n, l, k \geq 0, k' \neq k''$,

$$h((\gamma^* \gamma)^k) = \frac{1 - \mu^2}{1 - \mu^{2k+2}}, \quad h(\alpha^m \gamma^{*n} \gamma^l) = 0, \quad h(\alpha^{*m} \gamma^{*n} \gamma^l) = 0, \quad h(\gamma^{*k'} \gamma^{*k''}) = 0. \quad (1.2.15)$$

(Co)- representations of $SU_\mu(2)$

For each n in $\{0, 1/2, 1, \dots\}$, there is a unique irreducible representation T^n of dimension $2n + 1$. Denote by t_{ij}^n the ij th entry of T^n . They form an orthogonal basis of \mathcal{H} . Denote by e_{ij}^n the normalized t_{ij}^n s so that $\{e_{ij}^n : n = 0, 1/2, 1, \dots, i, j = -n, -n + 1, \dots, n\}$ is an orthonormal basis.

We recall from [37] that

$$t_{-1/2, -1/2}^{1/2} = \alpha, \quad t_{-1/2, 1/2}^{1/2} = -\mu \gamma^*, \quad t_{1/2, -1/2}^{1/2} = \gamma, \quad t_{1/2, 1/2}^{1/2} = \alpha^*. \quad (1.2.16)$$

Moreover, if we define

$$f_{n,i} = a(n, i) \alpha^{n-i} \gamma^{*n+i} \quad (1.2.17)$$

where $a(n, i)$ s are some constants as in [37], then $\{f_{n,i} : n = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, -n \leq i \leq n\}$ is an orthonormal basis of $SU_\mu(2)$ and $\Delta(f_{n,i}) = \sum_{k=-n}^n f_{n,k} \otimes t_{k,i}^n$.

The following recursive relations will be useful to us.

Proposition 1.2.27.

$$\begin{aligned}
& t_{i,l+1/2}^{l+1/2} \\
&= c_{11}(i, l) \gamma^* t_{i+1/2,l}^l + c_{12}(i, l) \alpha^* t_{i-1/2,l}^l \quad -l+1/2 \leq i \leq l-1/2, \\
&= c_{21}(i, l) \gamma^* t_{i+1/2,l}^l \quad i = -l-1/2, \\
&= c_{31}(i, l) \alpha^* t_{i-1/2,l}^l \quad i = l+1/2,
\end{aligned} \tag{1.2.18}$$

and for $j \leq l$,

$$\begin{aligned}
& t_{i,j}^{l+1/2} \\
&= c(l, i, j) \alpha t_{i+1/2,j+1/2}^l + c'(l, i, j) \gamma t_{i-1/2,j+1/2}^l \quad -l+1/2 \leq i \leq l-1/2, \\
&= d(l, j) \alpha t_{-l,j+1/2}^l + d'(l, j) \gamma^* t_{-l,j-\frac{1}{2}}^l \quad i = -l-1/2, \quad -l+\frac{1}{2} \leq j \leq l-\frac{1}{2}, \\
&= d''(l, j) \alpha t_{i+1/2,j+1/2}^l \quad i = -l-1/2, \quad j = -l-\frac{1}{2}, \\
&= e(l, j) \gamma t_{i-1/2,j+1/2}^l + e'(l, j) \alpha^* t_{i-\frac{1}{2},j-\frac{1}{2}}^l \quad i = l+1/2,
\end{aligned} \tag{1.2.19}$$

where $C_{pq}(il), c(l, i, j), d(l, j), d'_{l,j}, d''(l, j), e(l, j), e'(l, j)$ are all complex numbers.

Proof : It can be easily seen that

$$f_{l+\frac{1}{2},i} = c(l, i) \alpha f_{l,i+\frac{1}{2}} \tag{1.2.20}$$

for some constants $c(l, i)$.

Moreover, from (1.2.17) we have $\gamma^* f_{l,i} = a'(l, i) \alpha^{l-i} \gamma^{*l+i+1}$ for some constant $a'(l, i)$. This means that

$$\gamma^* f_{l,i} = \frac{a'(l, i)}{a(l+\frac{1}{2}, i+\frac{1}{2})} f_{l+\frac{1}{2},i+\frac{1}{2}}. \tag{1.2.21}$$

We have $f_{l+\frac{1}{2},l+\frac{1}{2}} = a(l+\frac{1}{2}, l+\frac{1}{2}) \gamma^{*2l+1}$ and $f_{l,l} = a(l, l) \gamma^{*2l}$ which implies that

$$f_{l+\frac{1}{2},l+\frac{1}{2}} = \frac{a(l+\frac{1}{2}, l+\frac{1}{2})}{a(l, l)} \gamma^* f_{l,l}. \tag{1.2.22}$$

Now, we proceed to prove (1.2.18). Applying coproduct on (1.2.22) and using (

1.2.17) and (1.2.21), we have

$$\begin{aligned}
& \sum_{k=-(l+\frac{1}{2})}^{l+\frac{1}{2}} f_{l+\frac{1}{2},k} \otimes t_{k,l+\frac{1}{2}}^{l+\frac{1}{2}} \\
&= \frac{a(l+\frac{1}{2}, l+\frac{1}{2})}{a(l,l)} (\gamma^* \otimes \alpha^* + \alpha \otimes \gamma^*) \left(\sum_{k=-l}^l f_{l,k} \otimes t_{k,l}^l \right) \\
&= \frac{a(l+\frac{1}{2}, l+\frac{1}{2})}{a(l,l)} \left[\sum_{k=-l}^l \gamma^* f_{l,k} \otimes \alpha^* t_{k,l}^l + \sum_{k=-l}^l \alpha f_{l,k} \otimes \gamma^* t_{k,l}^l \right] \\
&= \frac{a(l+\frac{1}{2}, l+\frac{1}{2})}{a(l,l)} \left[\sum_{k=-l}^l \frac{a'(l,k) f_{l+\frac{1}{2},k+\frac{1}{2}} \otimes \alpha^* t_{k,l}^l}{a(l+\frac{1}{2}, k+\frac{1}{2})} + \sum_{k=-l}^l \frac{f_{l+\frac{1}{2},k-\frac{1}{2}} \otimes \gamma^* t_{k,l}^l}{c(l,k-\frac{1}{2})} \right] \\
&= \frac{a(l+\frac{1}{2}, l+\frac{1}{2})}{a(l,l)} \left[\sum_{k=-l+\frac{1}{2}}^{l+\frac{1}{2}} \frac{a'(l,k-\frac{1}{2}) f_{l+\frac{1}{2},k} \otimes \alpha^* t_{k-\frac{1}{2},l}^l}{a(l+\frac{1}{2}, k)} + \sum_{k=-l-\frac{1}{2}}^{l-\frac{1}{2}} \frac{f_{l+\frac{1}{2},k} \otimes \gamma^* t_{k+\frac{1}{2},l}^l}{c(l,k)} \right].
\end{aligned}$$

Let $-l+\frac{1}{2} \leq k \leq l-\frac{1}{2}$. Then comparing coefficient of $f_{l+\frac{1}{2},k}$ we have $t_{k,l+\frac{1}{2}}^{l+\frac{1}{2}} = \frac{a(l+\frac{1}{2}, l+\frac{1}{2})}{a(l,l)} [\frac{a'(l,k-\frac{1}{2})}{a(l+\frac{1}{2},k)} \alpha^* t_{k-\frac{1}{2},l}^l + \frac{1}{c(l,k)} \gamma^* t_{k+\frac{1}{2},l}^l]$ which proves the first equation of (1.2.18).

Applying the same procedure for $k = -l-\frac{1}{2}$, we have $t_{k,l+\frac{1}{2}}^{l+\frac{1}{2}} = \frac{a(l+\frac{1}{2}, l+\frac{1}{2})}{a(l,l)} [\frac{1}{c(l,k)} \gamma^* t_{k+\frac{1}{2},l}^l]$ which proves the second equation of (1.2.18).

Similarly, for $k = l+\frac{1}{2}$, we have $t_{k,l+\frac{1}{2}}^{l+\frac{1}{2}} = \frac{a(l+\frac{1}{2}, l+\frac{1}{2})}{a(l,l)} [\frac{a'(l,k-\frac{1}{2})}{a(l+\frac{1}{2},k)} \alpha^* t_{k-\frac{1}{2},l}^l]$ which proves the third equation of (1.2.18). This completes the proof of (1.2.18).

Next, to prove (1.2.19), we apply coproduct on (1.2.20) and use (1.2.17) and (1.2.21) to have

$$\begin{aligned}
& \sum_{k=-(l+\frac{1}{2})}^{l+\frac{1}{2}} f_{l+\frac{1}{2},k} \otimes t_{k,i}^{l+\frac{1}{2}} \\
&= c(l,i) (\alpha \otimes \alpha - \mu \gamma^* \otimes \gamma) \left(\sum_{k=-l}^l f_{l,k} \otimes t_{k,i+\frac{1}{2}}^l \right) \\
&= c(l,i) \left(\sum_{k=-l}^l \alpha f_{l,k} \otimes \alpha t_{k,i+\frac{1}{2}}^l - \sum_{k=-n}^n \gamma^* f_{l,k} \otimes \mu \gamma t_{k,i+\frac{1}{2}}^l \right) \\
&= c(l,i) \sum_{k=-l}^l \frac{1}{c(l,k-\frac{1}{2})} f_{l+\frac{1}{2},k-\frac{1}{2}} \otimes \alpha t_{k,i+\frac{1}{2}}^l - c(l,i) \sum_{k=-l}^l \frac{a'(l,k)}{a(l+\frac{1}{2}, k+\frac{1}{2})} f_{l+\frac{1}{2},k+\frac{1}{2}} \otimes \mu \gamma t_{k,i+\frac{1}{2}}^l
\end{aligned}$$

$$\begin{aligned}
&= c(l, i) \sum_{k=-l-\frac{1}{2}}^{l-\frac{1}{2}} \frac{1}{c(l, k)} f_{l+\frac{1}{2}, k} \otimes \alpha t_{k+\frac{1}{2}, i+\frac{1}{2}}^l - c(l, i) \sum_{k=-l+\frac{1}{2}}^{l+\frac{1}{2}} \frac{a(l, k-\frac{1}{2})}{a(l+\frac{1}{2}, k)} f_{l+\frac{1}{2}, k} \otimes \\
&\mu \gamma t_{k-\frac{1}{2}, i+\frac{1}{2}}^l. \\
&\text{For } -l + \frac{1}{2} \leq k \leq l - \frac{1}{2}, \text{ by comparing coefficient of } f_{l+\frac{1}{2}, k} \text{ we have } t_{k, i}^{l+\frac{1}{2}} = \\
&\frac{c(l, i)}{c(l, k)} \alpha t_{k+\frac{1}{2}, i+\frac{1}{2}}^l - \frac{c(l, i) a(l, k-\frac{1}{2})}{a(l+\frac{1}{2}, k)} \mu \gamma t_{k-\frac{1}{2}, i+\frac{1}{2}}^l \text{ which proves the first equation of (1.2.19).} \\
&\text{Comparing coefficient of } f_{l+\frac{1}{2}, -l-\frac{1}{2}}, \text{ we have } t_{-l-\frac{1}{2}, i}^{l+\frac{1}{2}} = \frac{c(l, i)}{c(l, -l-\frac{1}{2})} \alpha t_{-l, i+\frac{1}{2}}^l \text{ from which} \\
&\text{we get the second and the third equation of (1.2.19).} \\
&\text{Comparing coefficient of } f_{l+\frac{1}{2}, l+\frac{1}{2}}, \text{ we have } t_{l+\frac{1}{2}, i}^{l+\frac{1}{2}} = -\frac{c(l, i) a(l, l)}{a(l+\frac{1}{2}, l+\frac{1}{2})} \mu \gamma t_{l, i+\frac{1}{2}}^l \text{ from which} \\
&\text{we get the last equation of (1.2.19).} \quad \square
\end{aligned}$$

We recall the following multiplication rule from Page 74, [37] which we are going to need :

$$t_{i, j}^l t_{i', j'}^{1/2} = \sum_{k=|l-1/2|, \dots, l+1/2} c_k(l, i, j, i', j') t_{i+i', j+j'}^k \quad (1.2.23)$$

($c_k(l, i, j, i', j')$ are scalars).

1.2.5 The Hopf *-algebras $\mathcal{O}(SU_\mu(2))$ and $\mathcal{U}_\mu(su(2))$

We define the Hopf *-algebra $\mathcal{O}(SU_\mu(2))$ following the notations of [37].

$\mathcal{O}(SL_\mu(2))$ is the complex associative algebra with generators a, b, c, d such that

$$ab = \mu ba, \quad ac = \mu ca, \quad bd = \mu db, \quad cd = \mu dc, \quad bc = cb, \quad ad - \mu bc = da - \mu^{-1} bc = 1. \quad (1.2.24)$$

The coproduct is given by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d.$$

The antipode is

$$\kappa(a) = d, \quad \kappa(b) = -b, \quad \kappa(c) = -c, \quad \kappa(d) = a$$

Finally, the counit is

$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0.$$

For all μ in \mathbb{R} , there is an involution of the algebra $\mathcal{O}(SL_\mu(2))$ determined by

$$a^* = d, \quad b^* = -\mu c. \quad (1.2.25)$$

The corresponding Hopf $*$ -algebra is denoted by $\mathcal{O}(SU_\mu(2))$.

Proposition 1.2.28. $\mathcal{O}(SU_\mu(2))$ can be identified with $(SU_\mu(2))_0$, i.e the Hopf $*$ -algebra generated by the matrix elements of irreducible unitary representations of $SU_\mu(2)$, via the isomorphism given on the generators by

$$\alpha \mapsto a, \gamma \mapsto c, \alpha^* \mapsto d, \gamma^* \mapsto -\mu^{-1}b. \quad (1.2.26)$$

Proof : $(SU_\mu(2))_0$ is generated by the matrix elements of the fundamental unitary of $SU_\mu(2)$, that is, the $*$ -algebra generated by α and γ . On the other hand, inserting (1.2.25) in (1.2.24), we have that $\mathcal{O}(SU_\mu(2))$ is generated by 4 elements a, b, c, d such that $ac = \mu ca$, $ac^* = \mu c^*a$, $cc^* = c^*c$, $a^*a + c^*c = 1$, $aa^* + \mu^2 c^*c = 1$. Comparing with the defining equations of $SU_\mu(2)$, that is, (1.2.10) - (1.2.14), it is clear that the above correspondence gives the required isomorphism. \square

Next, we recall from [51] the Hopf $*$ algebra $\mathcal{U}_\mu(su(2))$ which is the dual Hopf $*$ -algebra of $\mathcal{O}(SU_\mu(2))$. It is generated by elements F, E, K, K^{-1} with defining relations:

$$KK^{-1} = K^{-1}K = 1, KE = \mu EK, FK = \mu KF, EF - FE = (\mu - \mu^{-1})^{-1}(K^2 - K^{-2})$$

with involution $E^* = F$, $K^* = K$ and comultiplication :

$$\Delta(E) = E \otimes K + K^{-1} \otimes E, \Delta(F) = F \otimes K + K^{-1} \otimes F, \Delta(K) = K \otimes K.$$

The counit is given by $\epsilon(E) = \epsilon(F) = \epsilon(K - 1) = 0$ and antipode $\kappa(K) = K^{-1}$, $\kappa(E) = -\mu E$, $\kappa(F) = -\mu^{-1}F$.

There is a dual pairing $\langle \cdot, \cdot \rangle$ of $\mathcal{U}_\mu(su(2))$ and $\mathcal{O}(SU_\mu(2))$ given on the generators by :

$$\langle K^{\pm 1}, \alpha^* \rangle = \langle K^{\mp 1}, \alpha \rangle = \mu^{\pm \frac{1}{2}}, \langle E, \gamma \rangle = \langle F, -\mu \gamma^* \rangle = 1$$

and zero otherwise.

The left action \triangleright and right action \triangleleft of $\mathcal{U}_\mu(su(2))$ on $SU_\mu(2)$ are given by:

$f \triangleright x = \langle f, x_{(2)} \rangle x_{(1)}$, $x \triangleleft f = \langle f, x_{(1)} \rangle x_{(2)}$, $x \in \mathcal{O}(SU_\mu(2))$, $f \in \mathcal{U}_\mu(su(2))$ where we use the Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

The actions satisfy :

$$(f \triangleright x)^* = \kappa(f)^* \triangleright x^*, (x \triangleleft f)^* = x^* \triangleleft \kappa(f)^*, f \triangleright xy = (f_{(1)} \triangleright x)(f_{(2)} \triangleright y), xy \triangleleft f = (x \triangleleft f_{(1)})(y \triangleleft f_{(2)}).$$

The action on generators is given by :

$$\left\{ \begin{array}{llll} E \triangleright \alpha = -\mu\gamma^* & E \triangleright \gamma = \alpha^*, & E \triangleright \gamma^* = 0, & E \triangleright \alpha^* = 0, \\ F \triangleright (-\mu\gamma^*) = \alpha & F \triangleright \alpha^* = \gamma, & F \triangleright \alpha = 0, & F \triangleright \gamma = 0, \\ K \triangleright \alpha = \mu^{-\frac{1}{2}}\alpha, & K \triangleright (\gamma^*) = \mu^{\frac{1}{2}}\gamma^*, & K \triangleright \gamma = \mu^{-\frac{1}{2}}\gamma, & K \triangleright \alpha^* = \mu^{\frac{1}{2}}\alpha^*. \end{array} \right\} \quad (1.2.27)$$

$$\left\{ \begin{array}{llll} \gamma \triangleleft E = \alpha, & \alpha^* \triangleleft E = -\mu\gamma^*, & \alpha \triangleleft E = 0, & \gamma^* \triangleleft E = 0 \\ \alpha \triangleleft F = \gamma, & -\mu\gamma^* \triangleleft F = \alpha^*, & \gamma \triangleleft F = 0, & \alpha^* \triangleleft F = 0, \\ \alpha \triangleleft K = \mu^{-\frac{1}{2}}\alpha, & \gamma^* \triangleleft K = \mu^{-\frac{1}{2}}\gamma^*, & \gamma \triangleleft K = \mu^{\frac{1}{2}}\gamma, & \alpha^* \triangleleft K = \mu^{\frac{1}{2}}\alpha^*. \end{array} \right\} \quad (1.2.28)$$

1.2.6 The Wang algebras

Let us now recall the universal quantum groups as in [61], [59] and references therein. For an $n \times n$ positive invertible matrix $Q = (Q_{ij})$. let $A_{u,n}(Q)$ be the compact quantum group defined and studied in [60], [61], which is the universal C^* -algebra generated by $\{u_{kj}^Q, k, j = 1, \dots, n\}$ such that $u := ((u_{kj}^Q))$ satisfies

$$uu^* = I_n = u^*u, \quad u'Q\bar{u}Q^{-1} = I_n = Q\bar{u}Q^{-1}u'. \quad (1.2.29)$$

Here $u' = ((u_{ji}))$ and $\bar{u} = ((u_{ij}^*))$. The coproduct, say $\tilde{\Delta}$, is given by,

$$\tilde{\Delta}(u_{ij}) = \sum_{k=1}^n u_{ik}^Q \otimes u_{kj}^Q.$$

It may be noted that $A_{u,n}(Q)$ is the universal object in the category of compact quantum groups which admit an action on the finite dimensional C^* algebra $M_n(\mathbb{C})$ which preserves the functional $M_n \ni x \mapsto \text{Tr}(Q^T x)$, (see [63]) where the notion of a CQG and that of preservation of a functional by an action are as in subsection 1.2.7. We refer the reader to [61] for a detailed discussion on the structure and classification of such quantum groups.

Remark 1.2.29. *It was proved in [59] that in the case where $Q = I$, $\kappa(u_{ij}^I) = u_{ji}^{I*}$ and hence $\kappa^2 = \text{id}$ holds for $A_{u,n}(I)$.*

1.2.7 Action of a compact quantum group on a C^* algebra

We say that the compact quantum group (\mathcal{S}, Δ) (co)-acts on a unital C^* algebra \mathcal{B} , if there is a unital C^* -homomorphism (called an action) $\alpha : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{S}$ satisfying the following :

(bi) $(\alpha \otimes \text{id}) \circ \alpha = (\text{id} \otimes \Delta) \circ \alpha$, and

(bii) the linear span of $\alpha(\mathcal{B})(1 \otimes \mathcal{S})$ is norm-dense in $\mathcal{B} \otimes \mathcal{S}$.

It is known (see, for example, [60] , [44]) that (bii) is equivalent to the existence of a norm-dense, unital $*$ -subalgebra \mathcal{B}_0 of \mathcal{B} such that $\alpha(\mathcal{B}_0) \subseteq \mathcal{B}_0 \otimes_{\text{alg}} \mathcal{S}_0$ and on \mathcal{B}_0 , $(\text{id} \otimes \epsilon) \circ \alpha = \text{id}$.

We shall sometimes say that α is a ‘topological’ or C^* action to distinguish it from a normal action of von Neumann algebraic quantum group.

Definition 1.2.30. Let (\mathcal{S}, α) has a C^* action α on the C^* algebra \mathcal{B} . We say that the action α is **faithful** if there is no proper Woronowicz C^* -subalgebra \mathcal{S}_1 of \mathcal{S} such that α is a C^* action of \mathcal{S}_1 on \mathcal{B} .

Definition 1.2.31. Let (\mathcal{S}, α) has a C^* action α on the C^* algebra \mathcal{B} . A continuous linear functional ϕ on \mathcal{B} is said to be **invariant under α** if

$$(\phi \otimes \text{id})\alpha(b) = \phi(b).1_{\mathcal{S}}.$$

Now, we recall the work of Shuzhou Wang done in [60]. One can also see [4], [5].

The **quantum permutation group** \mathcal{QU}_n is defined to be the C^* algebra generated by a_{ij} ($i, j = 1, 2, \dots, n$) satisfying the following relations:

$$a_{ij}^2 = a_{ij} = a_{ij}^*, \quad i, j = 1, 2, \dots, n,$$

$$\sum_{j=1}^n a_{ij} = 1, \quad i = 1, 2, \dots, n,$$

$$\sum_{i=1}^n a_{ij} = 1, \quad i = 1, 2, \dots, n.$$

The name comes from the fact that the universal commutative C^* algebra generated by the above set of relations is isomorphic to $C(S_n)$ where S_n denotes the permutation group on n symbols.

Let us consider the category with objects as compact groups acting on on a n -point set $X_n = \{x_1, x_2, \dots, x_n\}$. If two groups G_1 and G_2 have actions α_1 and α_2 respectively, then a morphism from G_1 to G_2 is a group homomorphism ϕ such that $\alpha_2(\phi \times \text{id}) = \alpha_1$. Then $C(S_n)$ is the universal object in this category. It is proved in [60] that the quantum permutation group enjoys a similar property.

We have that $C(X_n) = C^*\{e_i : e_i^2 = e_i = e_i^*, \sum_{r=1}^n e_r = 1, i = 1, 2, \dots, n\}$. Then \mathcal{QU}_n has a C^* action on $C(X_n)$ via the formula:

$$\alpha(e_j) = \sum_{i=1}^n e_i \otimes a_{ij}, \quad j = 1, 2, \dots, n.$$

Proposition 1.2.32. *Consider the category with objects as CQG s having a C^* action on $C(X_n)$ and morphisms as CQG morphisms intertwining the actions as above. Then \mathcal{QU}_n is the universal object in this category.*

Now we note down a simple fact for future use.

Lemma 1.2.33. *Let α be an action of a CQG \mathcal{S} on $C(X)$ where X is a finite set. Then α automatically preserves the functional τ corresponding to the counting measure:*

$$(\tau \otimes \text{id})(\alpha(f)) = \tau(f) \cdot 1_{\mathcal{S}}.$$

Proof:

Let $X = \{1, \dots, n\}$ for some $n \in \mathbb{N}$ and denote by δ_i the characteristic function of the point i . Let $\alpha(\delta_i) = \sum_j \delta_j \otimes q_{ij}$ where $\{q_{ij} : i, j = 1 \dots n\}$ are the images of the canonical generators of the quantum permutation group as above. Then τ -preservation of α follows from the properties of the generators of the quantum permutation group, which in particular imply that $\sum_j q_{ij} = 1 = \sum_i q_{ij}$. \square

Wang also identified the universal object in the category of all CQG s having a C^* action α_1 on $M_n(\mathbb{C})$ (with morphisms as before) such that the functional $\frac{1}{n}\text{Tr}$ is kept invariant under α_1 . However, no such universal object exists if the invariance of the functional is not assumed. The precise statement is contained in the following theorem.

Before that, we recall that $M_n(\mathbb{C}) = C^*\{e_{ij} : e_{ij}e_{kl} = \delta_{jk}e_{il}, e_{ij}^* = e_{ji}, \sum_{r=1}^n e_{rr} = 1, i, j, k, l = 1, 2, \dots, n\}$.

Proposition 1.2.34. *Let $\mathcal{QU}_{M_n(\mathbb{C}), \frac{1}{n}\text{Tr}}$ be the C^* algebra with generators a_{ij}^{kl} and the following defining relations:*

$$\sum_{v=1}^n a_{ij}^{kv} a_{rs}^{vl} = \delta_{jr} a_{is}^{kl}, \quad i, j, k, l, r, s = 1, 2, \dots, n,$$

$$\sum_{v=1}^n a_{lv}^{sr} a_{vk}^{ji} = \delta_{jr} a_{lk}^{si}, \quad i, j, k, l, r, s = 1, 2, \dots, n,$$

$$a_{ij}^{kl*} = a_{ji}^{lk}, \quad i, j, k, l = 1, 2, \dots, n,$$

$$\sum_{r=1}^n a_{rr}^{kl} = \delta_{kl}, \quad k, l = 1, 2, \dots, n,$$

$$\sum_{r=1}^n a_{kl}^{rr} = \delta_{kl}, \quad k, l = 1, \dots, n.$$

Then,

(1) $\mathcal{QU}_{M_n(\mathbb{C}), \frac{1}{n}\text{Tr}}$ is a CQG with coproduct Δ defined by $\Delta(a_{ij}^{kl}) = \sum_{r,s=1}^n a_{rs}^{kl} \otimes a_{ij}^{rs}$, $i, j, k, l = 1, 2, \dots, n$.

(2) $\mathcal{QU}_{M_n(\mathbb{C}), \frac{1}{n}\text{Tr}}$ has a C^* action α_1 on $M_n(\mathbb{C})$ given by $\alpha_1(e_{ij}) = \sum_{k,l=1}^n e_{kl} \otimes a_{ij}^{kl}$, $i, j = 1, 2, \dots, n$. Moreover, $\mathcal{QU}_{M_n(\mathbb{C}), \frac{1}{n}\text{Tr}}$ is the universal object in the category of all CQG s having C^* action on $M_n(\mathbb{C})$ such that the functional $\frac{1}{n}\text{Tr}$ is kept invariant under the action.

(3) There does not exist any universal object in the category of all CQG s having C^* action on $M_n(\mathbb{C})$.

Proposition 1.2.35. *Since, any faithful state on a finite dimensional C^* algebra \mathcal{A} is of the form $\text{Tr}(Rx)$ for some operator R , it follows from Theorem 6.1, (2) of [60] that the universal CQG acting on any finite dimensional C^* algebra preserving a faithful state ϕ exists and is going to be denoted by $\mathcal{QU}_{\mathcal{A}, \phi}$.*

Notations:

We conclude this section on quantum groups by fixing some notations which will be used throughout this thesis. In particular, given a compact quantum group (\mathcal{S}, Δ) , the dense unital Hopf $*$ -subalgebra of \mathcal{S} generated by the matrix elements of the irreducible unitary representations will be denoted by \mathcal{S}_0 . Moreover, given an action $\gamma : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{S}$ of the compact quantum group (\mathcal{S}, Δ) on a unital C^* -algebra \mathcal{B} , the dense, unital $*$ -subalgebra of \mathcal{B} on which the action becomes an action by the Hopf $*$ -algebra \mathcal{S}_0 will be denoted by \mathcal{B}_0 . We shall use the Sweedler convention of abbreviating $\gamma(b) \in \mathcal{B}_0 \otimes_{\text{alg}} \mathcal{S}_0$ by $b_{(1)} \otimes b_{(2)}$, for b in \mathcal{B}_0 . This applies in particular to the canonical action of the quantum group \mathcal{S} on itself, by taking $\gamma = \Delta$.

Moreover, for a linear functional f on \mathcal{S} and an element c in \mathcal{S}_0 we recall the ‘convolution’ maps $f \triangleleft c := (f \otimes \text{id})\Delta(c)$ and $c \triangleright f := (\text{id} \otimes f)\Delta(c)$. We also define convolution of two functionals f and g by $(f \diamond g)(c) = (f \otimes g)(\Delta(c))$.

1.3 Rieffel deformation

In this section, we recall the notions of Rieffel’s formulation of deformation quantization ([46]) as well as Rieffel type deformation of CQG s due to Rieffel and Wang (as in [47] and [62]).

We begin with Rieffel deformation (as in [46]) from action of \mathbb{R}^n on a C^* algebra. In the following discussion and henceforth, the symbol $e(x)$ will stand for $e^{2\pi i x}$. Let V be a real vector space of dimension n and α be its strongly continuous isometric action on a complex Frechet space \mathcal{A} . Let $\{\|\cdot\|_j\}$ denote the family of seminorms which determine the topology of \mathcal{A} . It is assumed that α is isometric for each of the given seminorms on \mathcal{A} .

Let \mathcal{A}^∞ denote the space of smooth vectors for the action α , that is, $\mathcal{A}^\infty = \{a \in \mathcal{A} : v \rightarrow \alpha_v(a) \text{ is } C^\infty\}$.

Let $\{X_1, X_2, \dots, X_n\}$ be a basis of V and δ_k denotes the operator of partial differentiation on \mathcal{A}^∞ in the direction of X_k .

For any multi index $\mu = (\mu_1, \mu_2, \dots, \mu_n)$, we will let $\delta^\mu = \delta^{\mu_1} \delta^{\mu_2} \dots \delta^{\mu_n}$, $\mu! = \mu_1! \dots \mu_n!$, $|\mu| = \sum_{i=1}^n \mu_i$. We will equip \mathcal{A}^∞ with the semi norms: $\|a\|_{jk} = \sup_{i \leq j} \sum_{|\mu| \leq k} \frac{\|\delta^\mu a\|_i}{\mu!}$.

Let $C_b(V, \mathcal{A})$ denote the Frechet space of continuous bounded functions from V to \mathcal{A} , equipped with the semi norms $\|f\|_k = \sup_{v \in V} \|f(v)\|_k$. There is also a natural action of V on $C_b(V, \mathcal{A})$ by translation and let $C_u(V, \mathcal{A})$ denote the largest subspace of $C_b(V, \mathcal{A})$ on which this action is strongly continuous and let $\mathcal{B}^{\mathcal{A}}(V)$ denote the space of smooth vectors with respect to this action.

Let $W = V \times V$. Then $\mathcal{B}^{\mathcal{A}}(W)$ makes sense and for F in $\mathcal{B}^{\mathcal{A}}(W)$, one can define the oscillatory integral $\int \int F(u, v) e(u.v) du dv$ (where $u.v$ denote the usual inner product) in the following way:

We choose a basis of W and let L denote the lattice of points of W which have integer co-ordinates w.r.t this basis. Moreover, choose a positive ϕ_0 in $C_c^\infty(W)$ such that $\Phi = \sum_{p \in L} \phi_p$ vanishes nowhere on W where ϕ_p denotes the translate of ϕ by p belonging to L . Let $\phi = \frac{\phi_0}{\Phi}$.

It can be shown ([46]) that for F in $\mathcal{B}^{\mathcal{A}}(W)$, $\sum_{p \in L} \int (F \phi_p)(u, v) e(u.v) du dv$ converges absolutely in \mathcal{A} and $\int \int F(u, v) e(u.v) du dv$ is defined to be this sum. Moreover, this sum is independent of the choice of lattice and of ϕ . Thus,

$$\int \int F(u, v) e(u.v) du dv = \sum_{p \in L} \int (F \phi_p)(u, v) e(u.v) du dv. \quad (1.3.1)$$

For more details of oscillatory integral, we refer to [46] and references therein.

We will need the following results from [46].

Proposition 1.3.1. (Corollary 1.12, [46]) *Let F be a function in $\mathcal{B}^{\mathcal{A}}(V \times V)$ which depends only on the first variable, so that it is essentially an element of $\mathcal{B}^{\mathcal{A}}(V)$. Then $\int \int F(u) e(u.v) du dv = F(0)$. The same is true if instead F depends only on the second variable.*

Proposition 1.3.2. (Proposition 1.14, [46]) *Let S be a continuous linear transformation from \mathcal{A} into a Frechet space C . Let F belongs to $\mathcal{B}^{\mathcal{A}}(W)$. Then $S \circ F$ belongs to $\mathcal{B}^C(W)$ and $S(\int \int F(u, v) e(u.v) du dv) = \int \int S(F(u, v)) e(u.v) du dv$.*

Now, let \mathcal{A} be a Frechet algebra. Fix a skew symmetric matrix J on V . Then for all a, b in \mathcal{A}^∞ , $\alpha_{Ju}(a) \alpha_v(b)$ belongs to $\mathcal{B}^{\mathcal{A}}(W)$ and a new product \times_J is defined on \mathcal{A}^∞ by declaring $a \times_J b = \int_V \int_V \alpha_{Ju}(a) \alpha_v(b) e(u.v) du dv$.

If the Frechet algebra has an involution $*$ which is continuous and if α acts by $*$ automorphisms, then $*$ is also an involution for the deformed product \times_J .

Proposition 1.3.3. (Lemma 2.20, [46]) Let f, g belongs to \mathcal{B}^A and let g have the lattice L as a period lattice so that g can be viewed as a smooth function on the compact group $H = V/L$. Then $\int \int f(u)g(v)e(u.v)dudv = \sum_L f(p)(\int_H g(v)e(p.v)dv)$. A similar statement holds if instead it is f which is periodic.

Corollary 1.3.4.

$$\int \int e(\theta z_1)e(z_2)e(z_1.z_2)dz_1dz_2 = e(-\theta).$$

Proof : We have,

$$\begin{aligned} & \int \int e(\theta z_1)e(z_2)e(z_1.z_2)dz_1dz_2 \\ &= \sum_{p \in \mathbf{Z}} e(\theta p) \left(\int_{S^1} e(z_2)e(p.z_2)dz_2 \right) \\ &= \sum_{p \in \mathbf{Z}} e(\theta p) \delta_{p,-1} \\ &= e(-\theta). \end{aligned}$$

□

Now, we will define the C^* algebra constructed by Rieffel corresponding to the data $(\mathcal{A}, V, \alpha, J)$ where \mathcal{A} is also assumed to be a C^* algebra and α a C^* automorphism.

Let \mathcal{S}^A be the space of \mathcal{A} valued smooth functions on V such that the product of their derivatives with any complex valued polynomials on V are bounded under the supremum norm of \mathcal{S}^A . Then \mathcal{S}^A is a pre Hilbert right \mathcal{A} module with \mathcal{A} valued inner product defined by

$$\langle f, g \rangle_A = \int_V f(v)^* g(v) dv,$$

for f, g belonging to \mathcal{S}^A .

Then, for a in \mathcal{A} , one defines the operator $L_{\tilde{a}}$ on \mathcal{S}^A by

$$L_{\tilde{a}}(f)(x) = \int_V \int_V \alpha_{x+Ju}(a) f(x+v) e(u.v) dudv,$$

where f belongs to \mathcal{S}^A . Then

Proposition 1.3.5. (Theorem 4.6 of [46]) $L_{\tilde{a}}$ is a bounded operator having an adjoint on the pre Hilbert module \mathcal{S}^A and $a \mapsto L_{\tilde{a}}$ is a $*$ representation of the algebra $(\mathcal{A}^\infty, \times_J)$ into the C^* algebra of bounded operators on \mathcal{S}^A .

Now, by defining

$$\|a\|_J = \|L_{\tilde{a}}\|,$$

we have a pre- C^* norm $\|\cdot\|_J$ on \mathcal{A}^∞ endowed with the new product \times_J .

The completion of this pre C^* algebra is the deformed C^* algebra and is denoted by \mathcal{A}_J .

One has a natural Frechet topology on \mathcal{A}_J^∞ , given by a family of seminorms $\{\|\cdot\|_{n,J}\}$ where $\|a\|_{n,J} = \sum_{|\mu| \leq n} \frac{\|\alpha_{X^\mu}(a)\|_J}{\mu!}$

We recall the following Proposition from [46].

Proposition 1.3.6. (Proposition 4.10, [46]) Let J be fixed. Then for large enough k there is a constant c_k such that for all a in \mathcal{A}^∞ , we have $\|a\|_J \leq c_k \|a\|_{2k}$.

Proposition 1.3.7. (Proposition 7.1, [46]) Let α be an action of V on the C^* algebra \mathcal{A} , with \mathcal{A}^∞ its subalgebra of smooth vectors. Let J be a skew-symmetric operator on V , and let α also denote the corresponding action of V on \mathcal{A}_J . Then the subalgebra of smooth vectors in \mathcal{A}_J for α is exactly \mathcal{A}^∞ . Moreover, $(\mathcal{A}_J)_{-J} \cong \mathcal{A}$.

Corollary 1.3.8. \mathcal{A}^∞ and \mathcal{A}_J^∞ coincide as topological (Frechet) spaces.

Proof : The proof is essentially contained in the proof of Proposition 7.1 in [46] (Proposition 1.3.7 above). By Proposition 1.3.6, we know that there is a constant c_k such that for any a in \mathcal{A}^∞ and for any μ ,

$$\|X^\mu a\|_J \leq c_k \|X^\mu a\|_{2k} \leq c'_k \|a\|_j$$

for $j = |\mu| + 2k$ and a new constant c'_k . Thus, the inclusion of \mathcal{A}^∞ into \mathcal{A}_J^∞ is continuous for their Frechet topologies. Similarly, using $(\mathcal{A}_J)_{-J} = \mathcal{A}$, we deduce that the inclusion of \mathcal{A}_J^∞ into \mathcal{A}^∞ is continuous. This proves the result. \square

Examples

The Noncommutative Torus

Let $\mathcal{A} = C(\mathbb{T}^n)$. For $v = (v_1, v_2, \dots, v_n)$ in \mathbb{R}^n , $x = (x_1, x_2, \dots, x_n)$ in \mathbb{T}^n , f in $C(\mathbb{T}^n)$, the action α of \mathbb{R}^n on \mathcal{A} is given by $\alpha_v f(x) = f(x_1 e(v_1), x_2 e(v_2), \dots, x_n e(v_n))$. Let θ be a $n \times n$ skew symmetric matrix and $J = \frac{\theta}{2}$. Then \mathcal{A}_J can be seen to be equal to the noncommutative n tori \mathbb{T}_θ^n , that is the universal C^* algebra generated by unitaries U_i , $i = 1, 2, \dots, n$ satisfying $U_i U_j = e(\theta_{ij}) U_j U_i$ where θ_{ij} denotes the (i, j) th entry of the matrix θ . We will denote \mathbb{T}_θ^2 by the notation \mathcal{A}_θ .

The Rieffel deformed spheres

For a skew symmetric matrix θ , we recall from [19], the definition of S_θ^n .

Let $\mu, \nu = 1, 2, \dots, n$. Let $\lambda^{\mu\nu} = e^{i\theta_{\mu\nu}}$ where $\theta_{\mu\nu}$ is the (μ, ν) th entry of the matrix θ .

S_θ^{2n-1} is the universal C^* algebra generated by $2n$ elements z^μ, \bar{z}^μ with relations:

$$z^\mu z^\nu = \lambda^{\mu\nu} z^\nu z^\mu, \quad \bar{z}^\mu \bar{z}^\nu = \lambda^{\mu\nu} \bar{z}^\nu \bar{z}^\mu, \quad (1.3.2)$$

$$\bar{z}^\mu z^\nu = \lambda^{\nu\mu} z^\nu \bar{z}^\mu, \quad (1.3.3)$$

$$(z^\mu)^* = \bar{z}^\mu, \quad (1.3.4)$$

$$\sum_{\mu=1}^n z^\mu \bar{z}^\mu = 1. \quad (1.3.5)$$

It can be easily seen that S_θ^{2n-1} is obtained by the Rieffel deformation of $C(S^{2n-1})$ using the $2n \times 2n$ skew symmetric matrix J whose (μ, ν) th entry is $\frac{\lambda^{\mu,\nu}}{2}$ and the \mathbb{R}^{2n} action on $C(S^{2n-1})$ given by $\alpha_v f(x_1, \dots, x_{2n}) = f(x_1 e(v_1), \dots, x_{2n} e(v_{2n}))$ ($v = (v_1, \dots, v_{2n})$ is in \mathbb{R}^{2n} , f is in $C^\infty(S^{2n-1})$).

S_θ^{2n} is the universal C^* algebra generated by $2n+1$ elements $\{z^\mu, \bar{z}^\mu, x : \mu = 1, 2, \dots, n\}$ where z^μ, \bar{z}^μ satisfy (1.3.2) - (1.3.4), x is a self adjoint element satisfying the relations $xz^\mu = z^\mu x$ for all $\mu = 1, 2, \dots, n$ and $\sum_{\mu=1}^n z^\mu \bar{z}^\mu + x^2 = 1$.

S_θ^{2n} is the Rieffel deformation of $C(S^{2n})$ by the action of \mathbb{R}^{2n+1} on $C(S^{2n})$ similar to above and a $(2n+1) \times (2n+1)$ matrix J' such that $(J')_{\mu,\nu} = \frac{\lambda^{\mu,\nu}}{2}$ if $\mu \leq 2n, \nu \leq 2n$ and 0 otherwise.

1.3.1 Rieffel Deformation of compact quantum group

Here we describe the Rieffel deformation of a CQG as in [62].

Let (\mathcal{A}, Δ) be a CQG with $C(\mathbb{T}^n)$ as a quantum subgroup. Let π be the corresponding CQG morphism from \mathcal{A} to $C(\mathbb{T}^n)$.

Let η be the canonical homomorphism from \mathbb{R}^n to \mathbb{T}^n given by $\eta(x_1, x_2, \dots, x_n) = (e(x_1), e(x_2), \dots, e(x_n))$ and ev_x be the state on $C(\mathbb{T}^n)$ obtained by evaluation of a function at the point x in \mathbb{T}^n .

Now, put

$$\lambda_{\eta(s)} = (ev_{\eta(-s)} \pi \otimes \text{id}) \Delta, \quad (1.3.6)$$

$$\rho_{\eta(u)} = (\text{id} \otimes ev_{\eta(u)} \pi) \Delta. \quad (1.3.7)$$

We will use the notation $\Omega(u)$ for $ev_{\eta(u)} \pi$.

Then there is a \mathbb{R}^{2n} action on \mathcal{A} defined by

$$\chi_{(s,u)} = \lambda_{\eta(s)} \rho_{\eta(u)}. \quad (1.3.8)$$

Fix a skew symmetric matrix J on \mathbb{R}^n and put

$$\tilde{J} = J \oplus (-J).$$

Then, by the prescription of Rieffel as described above, we have a C^* algebra $\mathcal{A}_{\tilde{J}}$. Shuzhou Wang showed in [62] that $\mathcal{A}_{\tilde{J}}$ can be made into a CQG.

The $*$ -algebra generated by the matrix elements of unitary irreducible representations of \mathcal{A} (denoted by \mathcal{A}_0) is dense in the space \mathcal{A}^∞ of smooth vectors of the action χ under the Frechet topology and hence is dense in the C^* algebra $\mathcal{A}_{\tilde{J}}$ under the C^* norm of $\mathcal{A}_{\tilde{J}}$. On \mathcal{A}_0 , the Hopf $*$ -algebra structure remains unchanged and this extends to a CQG structure on $\mathcal{A}_{\tilde{J}}$.

We quote the following result (Remark 3.10 (2), [62]) which will be used later.

Proposition 1.3.9. *The Haar measure $h_{\tilde{J}}$ of $\mathcal{A}_{\tilde{J}}$ is still the same as the Haar measure on the common subspace \mathcal{A}_0 .*

Lemma 1.3.10. *The Haar state (say h) of \mathcal{A} coincides with the Haar state on $\mathcal{A}_{\tilde{J}}$ (say h_J) on the common subspace \mathcal{A}^∞ , and moreover, $h(a \times_{\tilde{J}} b) = h(ab)$ for a, b in \mathcal{A}^∞ .*

Proof : We recall (Proposition 1.3.9) that $h = h_J$ on \mathcal{A}_0 . By using $h(\Omega(-s) \otimes \text{id}) = \Omega(-s)(\text{id} \otimes h)$ and $h(\text{id} \otimes \Omega(u)) = \Omega(u)(h \otimes \text{id})$, we have for a in \mathcal{Q}_0 ,

$$\begin{aligned} h(\chi_{s,u}(a)) &= \Omega(-s)(\text{id} \otimes h)\Delta(\text{id} \otimes \Omega(u))\Delta(a) \\ &= \Omega(-s)(h((\text{id} \otimes \Omega(u))\Delta(a))1) \\ &= h((\text{id} \otimes \Omega(u))\Delta(a)) \\ &= \Omega(u)(h(a).1) \\ &= h(a). \end{aligned}$$

Therefore,

$$h_{\chi_{s,u}}(b) = h(b) \text{ for all } b \text{ in } \mathcal{Q}_0. \quad (1.3.9)$$

Now,

$$\begin{aligned} h(a \times_{\tilde{J}} b) &= \int \int h(\chi_{\tilde{J}u}(a)\chi_v(b))e(u.v)dudv \\ &= \int \int h(\chi_v(\chi_{\tilde{J}u-v}(a)b))e(u.v)dudv \\ &= \int \int h(\chi_t(a)b)e(s.t)dsdt, \end{aligned}$$

where $s = -u, t = \tilde{J}u - v$, which by Proposition 1.3.1 equals $h(\chi_0(a)b) = h(ab)$. That is, we have proved

$$\langle a, b \rangle_J = \langle a, b \rangle \quad \forall a, b \in \mathcal{Q}_0, \quad (1.3.10)$$

where $\langle \cdot, \cdot \rangle_J$ and $\langle \cdot, \cdot \rangle$ respectively denote the inner products of $L^2(h_J)$ and $L^2(h)$. We now complete the proof of the lemma by extending (1.3.10) from \mathcal{Q}_0 to \mathcal{Q}^∞ , by using the fact that \mathcal{Q}^∞ is a common subspace of the Hilbert spaces $L^2(h)$ and $L^2(h_J)$ and moreover, \mathcal{Q}_0 is dense in both these Hilbert spaces. In particular, taking $a = 1$ in \mathcal{Q}_0 , we have $h = h_J$ on \mathcal{Q}^∞ . \square

Remark 1.3.11. Lemma 1.3.10 implies in particular that for every fixed a_1, a_2 in \mathcal{Q}_0 , the functional $\mathcal{Q}_0 \ni b \mapsto h(a_1 \times_{\tilde{J}} b \times_{\tilde{J}} a_2) = h(b \times_{\tilde{J}} a_2 \times_{\tilde{J}} (f_1 \triangleleft a_1 \triangleright f_1))$ (where f_1 is as in Remark 1.2.20) $= h(b(a_2 \times_{\tilde{J}} (f_1 \triangleleft a_1 \triangleright f_1)))$ extends to a bounded linear functional on \mathcal{Q} .

Let e be the identity of \mathbb{T}^{2n} and U_n be a sequence of neighbourhoods of e shrinking to e , f_n smooth, positive functions with support contained inside U_n such that $\int_{\mathbb{T}^{2n}} f_n(z) dz = 1$ for all n .

Let us denote the action of \mathbb{T}^{2n} action on \mathcal{A}^∞ induced by χ by $\tilde{\chi}$. Define $\lambda_{f_n}(a) = \int_{\mathbb{T}^{2n}} \tilde{\chi}_z(a) f_n(z) dz$. Then, we have the following result:

Lemma 1.3.12. $\lambda_{f_n}(a)$ belongs to \mathcal{Q}^∞ and

$$\int_{\mathbb{T}^{2n}} \tilde{\chi}_z(a) f_n(z) dz \rightarrow a \text{ as } n \rightarrow \infty.$$

Proof : We note that, by using the translation invariance of Haar measure, for all g in \mathbb{T}^{2n} , $\tilde{\chi}_g(\lambda_{f_n}(a)) = \int_{\mathbb{T}^{2n}} f_n(g^{-1}h) \tilde{\chi}_h(a) dh$. Therefore, $\tilde{\chi}_g(\lambda_{f_n}(a)) - \lambda_{f_n}(a) = \int_{\mathbb{T}^{2n}} (f_n(g^{-1}h) - f_n(h)) \tilde{\chi}_h(a) dh$ which proves the first part.

Now we prove the second part. As $\int_{\mathbb{T}^{2n}} f_n(z) dz = 1$ for all n and $\text{supp}(f_n) \subseteq U_n$, we have

$$\begin{aligned} & \left\| \int_{\mathbb{T}^{2n}} \tilde{\chi}_z(a) f_n(z) dz - a \right\| \\ &= \left\| \int_{\mathbb{T}^{2n}} \tilde{\chi}_z(a) f_n(z) dz - a \int_{\mathbb{T}^{2n}} f_n(z) dz \right\| \\ &= \left\| \int_{\mathbb{T}^{2n}} (\tilde{\chi}_z(a) - a) f_n(z) dz \right\| \\ &= \left\| \int_{U_n} (\tilde{\chi}_z(a) - a) f_n(z) dz \right\|. \end{aligned}$$

Now, using the fact that the map $z \mapsto \tilde{\chi}_z(a)$ is continuous for all a , we deduce that for all $\epsilon > 0$, there exists n such that for all z in U_n , $\|\tilde{\chi}_z(a) - \tilde{\chi}_0(a)\| < \epsilon$, that is,

$$\|\tilde{\chi}_z(a) - a\| < \epsilon.$$

Hence, $\|\int_{\mathbb{T}^{2n}} \lambda_z(a) f_n(z) dz - a\| \leq \epsilon \int_{\mathbb{T}^{2n}} f_n(z) = \epsilon$ which proves the lemma. \square

Lemma 1.3.13. *If h is faithful on \mathcal{Q} , then h_J is faithful on $\mathcal{Q}_{\tilde{J}}$.*

Proof : Let $a \geq 0, \in \mathcal{Q}_{\tilde{J}}$ be such that $h_J(a) = 0$. Let λ_{f_n} be as defined above. Then,

$$\begin{aligned} h_J(\lambda_{f_n}(a)) &= \int_{\mathbb{T}^{2n}} h_J(\tilde{\chi}_z(a)) f_n(z) dz \\ &= \int_{\mathbb{T}^{2n}} h_J(a) f_n(z) dz \\ &\quad (\text{ by (1.3.9) }) \\ &= 0, \end{aligned}$$

so we have $h(\lambda_{f_n}(a)) = 0$, since h and h_J coincide on \mathcal{Q}^∞ by Lemma 1.3.10 and $\lambda_{f_n}(a)$ belongs to \mathcal{Q}^∞ .

Now we fix some notation which we are going to use in the rest of the proof. Let $L^2(h)$ and $L^2(h_J)$ denote the G.N.S spaces of \mathcal{Q} and $\mathcal{Q}_{\tilde{J}}$ respectively with respect to the Haar states. Let i and i_J be the canonical maps from \mathcal{Q} and $\mathcal{Q}_{\tilde{J}}$ to $L^2(h)$ and $L^2(h_J)$ respectively. Also, let Π_J denote the G.N.S representation of \mathcal{Q}_J . Using the facts $h(b^* \times_{\tilde{J}} b) = h(b^*b)$ for all b in \mathcal{Q}^∞ and $h = h_J$ on \mathcal{Q}^∞ (by Lemma 1.3.10), we get $\|i_J(b)\|_{L^2(h_J)}^2 = \|i(b)\|_{L^2(h)}^2$ for all b in \mathcal{Q}^∞ . So the map sending $i(b)$ to $i_J(b)$ is an isometry from a dense subspace of $L^2(h)$ onto a dense subspace of $L^2(h_J)$, hence it extends to a unitary, say $\Gamma : L^2(h) \rightarrow L^2(h_J)$. We also note that the maps i and i_J agree on \mathcal{Q}^∞ .

Now, $a \geq 0$ means that $\lambda_{f_n}(a)$ is positive in $\mathcal{Q}_{\tilde{J}}$ and therefore, $\lambda_{f_n}(a) = b^* \times_{\tilde{J}} b$ for some b in $\mathcal{Q}_{\tilde{J}}$. So $h(\lambda_{f_n}(a)) = 0$ implies $h_J(\lambda_{f_n}(a)) = 0$ and therefore $\|i_J(b)\|_{L^2(h_J)}^2 = 0$. Therefore, one has $\Pi_J(b^*)i_J(b) = 0$, and hence $i_J(b^*b) = i_J(\lambda_{f_n}(a)) = 0$. It thus follows that $\Gamma(i(\lambda_{f_n}(a))) = 0$, which implies $i(\lambda_{f_n}(a)) = 0$. But the faithfulness of h means that i is one one, hence $\lambda_{f_n}(a) = 0$ for all n . Thus, recalling Lemma 1.3.12, we have $a = \lim_{n \rightarrow \infty} \lambda_{f_n}(a) = 0$, which proves the faithfulness of h_J . \square

At this point, we note a useful implication of the Lemma 1.3.10. Let us make use of the identification of \mathcal{Q}_0 as a common vector-subspace of all $\mathcal{Q}_{\tilde{J}}$. To be precise, we shall sometimes denote this identification map from \mathcal{Q}_0 to $\mathcal{Q}_{\tilde{J}}$ by ρ_J .

Corollary 1.3.14. *Let W be a finite-dimensional (say, n -dimensional) unitary representation of \mathcal{Q} , with \tilde{W} belonging to $M_n(\mathbb{C}) \otimes \mathcal{Q}_0$ be the corresponding unitary. Then, for any J , we have that $\tilde{W}_J := (\text{id} \otimes \rho_J)(\tilde{W})$ is unitary in $\mathcal{Q}_{\tilde{J}}$, giving a unitary n -dimensional*

representation of $\mathcal{Q}_{\tilde{J}}$. In other words, any finite dimensional unitary representation of \mathcal{Q} is also a unitary representation of $\mathcal{Q}_{\tilde{J}}$.

Proof:

Since the coalgebra structures of \mathcal{Q} and $\mathcal{Q}_{\tilde{J}}$ are identical, and \widetilde{W}_J is identical with \widetilde{W} as a linear map, it is obvious that \widetilde{W}_J gives a nondegenerate representation of $\mathcal{Q}_{\tilde{J}}$. Let $y = (\text{id} \otimes h)(\widetilde{W}_J^* \widetilde{W}_J)$. It follows from the proof of Proposition 6.4 of [41] that y is invertible positive element of M_n and $(y^{\frac{1}{2}} \otimes 1) \widetilde{W}_J (y^{-\frac{1}{2}} \otimes 1)$ gives a unitary representation of $\mathcal{Q}_{\tilde{J}}$. We claim that $y = 1$, which will complete the proof of the corollary. For convenience, let us write W in the Sweedler notation: $W = w_{(1)} \otimes w_{(2)}$. We note that by Lemma 1.3.10, we have

$$\begin{aligned} & (\text{id} \otimes h)(\widetilde{W}_J^* \widetilde{W}_J) \\ &= w_{(1)}^* w_{(1)} h(w_{(2)}^* \times_{\tilde{J}} w_{(2)}) \\ &= w_{(1)}^* w_{(1)} h(w_{(2)}^* w_{(2)}) \\ &= (\text{id} \otimes h)(\widetilde{W}^* \widetilde{W}) = (\text{id} \otimes h)(1 \otimes 1) = 1. \end{aligned}$$

□

Example The Rieffel deformed orthogonal groups

Let θ be a $n \times n$ skew symmetric matrix. $C(\mathbb{T}^n)$ sits inside $C(O(n))$ as a quantum subgroup. It can be easily seen that $O_{\theta}(n)$ is obtained by Rieffel deformation from $C(O(n))$ by using the induced \mathbb{R}^{2n} action as given in the equation (1.3.8) and considering the matrix $\tilde{J} = -J \oplus J$ when n is even and $-J' \oplus J'$ when n is odd where J and J' are the matrices introduced while giving the definition of the θ deformed spheres.

1.4 Classical Riemannian geometry

In this section we recall some classical facts regarding manifolds which will be useful to us later on.

1.4.1 Classical Hilbert space of forms

Let M be an n dimensional Riemannian manifold and $\Omega^k(M)$ ($k = 0, 1, 2, \dots, n$) be the space of smooth k -forms. Set $\Omega^k(M) = \{0\}$ for $k > n$. The de-Rham differential d maps $\Omega^k(M)$ to $\Omega^{k+1}(M)$. Let $\Omega \equiv \Omega(M) = \oplus_k \Omega^k(M)$. We will denote the Riemannian volume element by $d\text{vol}$. We recall that the Hilbert space $L^2(M)$ is obtained by completing the space $\{f \in C_c^\infty(M)\}$ with respect to the pre-inner product given by $\langle f_1, f_2 \rangle = \int_M \overline{f_1} f_2 d\text{vol}$.

In an analogous way, one can construct a canonical Hilbert space of forms. The Riemannian metric $\langle \cdot, \cdot \rangle_m$ (for m in M) on $T_m M$ induces an inner product on the vector space $T_m^* M$ and hence also $\Lambda^k T_m^* M$, which will be again denoted by $\langle \cdot, \cdot \rangle_m$. This gives a natural pre-inner product on the space of compactly supported k -forms by integrating the compactly supported smooth function $m \mapsto \langle \omega(m), \eta(m) \rangle_m$ over M . We will denote the completion of this space by $\mathcal{H}^k(M)$. Let $\mathcal{H} = \oplus_k \mathcal{H}^k(M)$.

Then, one can view $d : \Omega \rightarrow \Omega$ as an unbounded, densely defined operator (again denoted by d) on the Hilbert space \mathcal{H} with the domain Ω . It can be verified that it is closable.

1.4.2 Isometry groups of classical manifolds

Let M be a Riemannian manifold of dimension n . Then the collection of all isometries of M has a natural group structure and is denoted by $ISO(M)$. Let C and U be respectively a compact and open subset of M and let $W(C, U) = \{h \in ISO(M) : h.C \subseteq U\}$. The compact open topology on $ISO(M)$ is the smallest topology on $ISO(M)$ for which the sets $W(C, U)$ are open. It follows (see [34]) that under this topology, $ISO(M)$ is a closed locally compact topological group. Moreover, if M is compact, $ISO(M)$ is also compact.

We recall that the Laplacian \mathcal{L} on M is an unbounded densely defined self adjoint operator $-d^*d$ on the space of zero forms $\mathcal{H}^0(D) = L^2(M, \text{dvol})$ which has the local expression

$$\mathcal{L}(f) = -\frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (g^{ij} \sqrt{\det(g)} \frac{\partial}{\partial x_i} f)$$

for f in $C^\infty(M)$ and where $g = ((g_{ij}))$ is the Riemannian metric and $g^{-1} = ((g^{ij}))$. It is well known that on a compact manifold, the Laplacian has compact resolvents. Thus, the set of eigenvalues of \mathcal{L} is countable, each having finite multiplicities, and accumulating only at infinity. Moreover, there exists an orthonormal basis of $L^2(M)$ consisting of eigenvectors of \mathcal{L} which belong to $C^\infty(M)$. It can be shown (Lemma 2.3 of [30]) that for a compact manifold, the complex linear span of the eigenvectors of \mathcal{L} is dense in $C^\infty(M)$ in the sup norm.

The following result is in the form in which it has been stated and proved in [30] (Proposition 2.1).

Proposition 1.4.1. *Let M be a compact Riemannian manifold. Let \mathcal{L} be the Laplacian of M . A smooth map $\gamma : M \rightarrow M$ is a Riemannian isometry if and only if γ commutes with \mathcal{L} in the sense that $\mathcal{L}(f \circ \gamma) = (\mathcal{L}(f)) \circ \gamma$ for all f in $C^\infty(M)$.*

Using this fact, we give an operator theoretic proof of the fact that for a compact manifold, $ISO(M)$ is compact. As the action of $ISO(M)$ commutes with the Laplacian,

it has a unitary representation on $L^2(M)$. As the action preserves the finite dimensional eigenspaces of the Laplacian, $ISO(M)$ is a subgroup of $U(d_1) \times U(d_2) \times \dots$ (where $\{d_i : i \geq 0\}$ denote the dimension of the eigenspaces of the Laplacian and $U(d)$ denotes the group of unitary operators on a Hilbert space of dimension d) which is a compact group. As $ISO(M)$ is closed, it is a closed subgroup of a compact group, hence compact. We will see that this technique can be generalized in the noncommutative set-up in the chapters 2 and 3.

Proposition 1.4.1 has the generalization in a more general context.

Let us fix some notations. Let Y be a compact metrizable space and $\theta : M \times Y \rightarrow M$. Let $\xi_y : M \rightarrow M$ defined by $\xi_y(m) = \theta(m, y)$. Let $\alpha : C(M) \rightarrow C(M) \otimes C(Y) \cong C(M \times Y)$ be defined by $\alpha(f)(m, y) = f(\theta(m, y))$ for all y in Y , m in M . For a state ϕ on $C(Y)$, denote by α_ϕ , the map: $(\text{id} \otimes \phi)\alpha : C(M) \rightarrow C(M)$. Lastly, let \mathcal{A}_0^∞ be the span of eigenvectors of the Laplacian \mathcal{L} of M .

Then, we have the following(Lemma 2.5 of [30]):

Proposition 1.4.2. *The following are equivalent:*

- a. *For every y in Y , ξ_y is smooth isometric.*
- b. *For every state ϕ on $C(Y)$, we have $\alpha_\phi(\mathcal{A}_0^\infty) \subseteq \mathcal{A}_0^\infty$ and $\alpha_\phi \mathcal{L} = \mathcal{L} \alpha_\phi$ on \mathcal{A}_0^∞ .*

Example 1.4.3. 1. *The isometry group of the n -sphere S^n is $O(n+1)$ where the action is given by the usual action of $O(n+1)$ on \mathbb{R}^{n+1} . The subgroup of $O(n+1)$ consisting of all orientation preserving isometries on S^n is $SO(n+1)$.*

2. *The isometry group of the circle S^1 is $S^1 \rtimes \mathbb{Z}_2$. Here the \mathbb{Z}_2 ($= \{0, 1\}$) action on S^1 is given by $1.z = \bar{z}$ where z is in S^1 while the action of S^1 is its action on itself.*

3. *$ISO(\mathbb{T}^n) \cong \mathbb{T}^n \rtimes (\mathbb{Z}_2^n \rtimes S_n)$ where S_n is the permutation group on n symbols. Here an element of S_n acts on an element $(z_1, z_2, \dots, z_n) \in \mathbb{T}^n$ by permutation. If the generator of i -th copy of \mathbb{Z}_2^n is denoted by 1_i , then the action of 1_i is given by $1_i(z_1, z_2, \dots, z_n) = (z_1, \dots, z_{i-1}, \bar{z}_i, z_{i+1}, \dots, z_n)$ where $(z_1, z_2, \dots, z_n) \in \mathbb{T}^n$. Lastly, the action of \mathbb{T}^n on itself is its usual action.*

1.4.3 Spin Groups and Spin manifolds

We begin with Clifford algebras. Let Q be a quadratic form on an n dimensional vector space V . Then $Cl(V, Q)$ will denote the universal associative algebra \mathcal{C} equipped with a linear map $i : V \rightarrow \mathcal{C}$, such that $i(V)$ generates \mathcal{C} as a unital algebra satisfying $i(V)^2 = Q(V).1$

Let $\beta : V \rightarrow Cl(V, Q)$ be defined by $\beta(x) = -i(x)$. Then, $Cl(V, Q) = Cl^0(V, Q) \oplus Cl^1(V, Q)$ where $Cl^0(V, Q) = \{x \in Cl(V, Q) : \beta(x) = x\}$, $Cl^1(V, Q) = \{x \in Cl(V, Q) : \beta(x) = -x\}$.

We will denote by \mathcal{C}_n and $\mathcal{C}_n^{\mathbb{C}}$ the Clifford algebras $Cl(\mathbb{R}^n, -x_1^2 - \dots - x_n^2)$ and $Cl(\mathbb{C}^n, z_1^2 + \dots + z_n^2)$ respectively.

We will denote the vector space $\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ by the symbol Δ_n . It follows that $\mathcal{C}_n^{\mathbb{C}} = \text{End}(\Delta_n)$ if n is even and equals $\text{End}(\Delta_n) \oplus \text{End}(\Delta_n)$ if n is odd. There is a representation $\mathcal{C}_n^{\mathbb{C}} \rightarrow \text{End}(\Delta_n)$ which is the isomorphism with $\text{End}(\Delta_n)$ when n is even and in the odd case, it is the isomorphism with $\text{End}(\Delta_n) \oplus \text{End}(\Delta_n)$ followed by the projection onto the first component. This representation restricts to \mathcal{C}_n , to be denoted by κ_n and called the spin representation. This representation is irreducible when n is odd and for n even, it decomposes into two irreducible representations which decomposes Δ_n into a direct sum of two vector spaces Δ_n^+ and Δ_n^- .

$\text{Pin}(n)$ is defined to be the subgroup of \mathcal{C}_n generated by elements of the form $\{x : \|x\| = 1, x \in \mathbb{R}^n\}$. $\text{Spin}(n)$ is the group given by $\text{Pin}(n) \cap \mathcal{C}_n^0$. There exists a continuous group homomorphism from $\text{Pin}(n)$ to $O(n)$ which restricts to a two covering map $\lambda : \text{Spin}(n) \rightarrow SO(n)$.

Let M be an n -dimensional orientable Riemannian manifold. Then we have the oriented orthonormal bundle of frames over M (which is a principal $SO(n)$ bundle) which we will denote by F .

Such a manifold M is said to be a **spin manifold** if there exists a pair (P, Λ) (called a spin structure) where

(1) P is a $\text{Spin}(n)$ principal bundle over M .

(2) Λ is a map from P to F such that it is a 2-covering as well as a bundle map over M .

(3) $\Lambda(p.\hat{g}) = \Lambda(p).g$ where $\lambda(\hat{g}) = g$.

Given such a spin structure, we consider the associated bundle $S = P \times_{\text{Spin}(n)} \Delta_n$ called the ‘ **bundle of spinors** ’.

1.4.4 Dirac operators

We follow the notations of the previous subsection. On the space of smooth sections of the bundle of spinors S , one can define an inner product by

$$\langle s_1, s_2 \rangle_S = \int_M \langle s_1(x), s_2(x) \rangle d\text{vol}(x)$$

The Hilbert space obtained by completing the space of smooth sections with respect to this inner product is denoted by $L^2(S)$ and its members are called square integrable spinors. The Levi Civita connection on M induces a canonical connection on S which we will denote by ∇^S .

Definition 1.4.4. *The Dirac operator on M is the self-adjoint extension of the fol-*

lowing operator D defined on the space of smooth sections of S :

$$(Ds)(m) = \sum_{i=1}^n \kappa_n(X_i(m))(\nabla_{X_i}^S s)(m),$$

where (X_1, \dots, X_n) are local orthonormal (with respect to the Riemannian metric) vector fields defined in a neighborhood of m . In this definition, we have viewed $X_i(m)$ belonging to $T_m(M)$ as an element of the Clifford algebra $Cl_{\mathbb{C}}(T_m M)$, hence $\kappa_n(X_i(m))$ is a map on the fibre of S at m , which is isomorphic with Δ_n . The self-adjoint extension of D is again denoted by the same symbol.

We recall three important facts about the Dirac operator:

Proposition 1.4.5. (1) $C^\infty(M)$ acts on S by multiplication and this action extends to a representation, say π , of the C^* algebra $C(M)$ on the Hilbert space $L^2(S)$.

(2) For f in $C^\infty(M)$, $[D, \pi(f)]$ has a bounded extension.

(3) Furthermore, the Dirac operator on a compact manifold has compact resolvents.

As the action of an element f in $C^\infty(M)$ on $L^2(S)$ is by multiplication operator, we will use the symbol M_f in place of $\pi(f)$.

The Dirac operator carries a lot of geometric and topological information. We give two examples.

(a) The Riemannian metric of the manifold is recovered by

$$d(p, q) = \sup_{\phi \in C^\infty(M), \|[D, M_\phi]\| \leq 1} |\phi(P) - \phi(q)|. \quad (1.4.1)$$

(b) For a compact manifold, the operator e^{-tD^2} is trace class for all $t > 0$. Then the volume form of the manifold can be recovered by the formula

$$\int_M f dvol = c(n) \lim_{t \rightarrow 0} \frac{\text{Tr}(M_f e^{-tD^2})}{\text{Tr}(e^{-tD^2})}$$

where $\dim M = n$, $c(n)$ is a constant depending on the dimension.

1.5 Noncommutative Geometry

In this section, we recall those basic concepts of noncommutative geometry which we are going to need. We refer to [17], [40], [22] for more details.

1.5.1 Spectral triples

Motivated by the facts in Proposition 1.4.5, Alain Connes defined his formulation of noncommutative manifold based on the idea of a spectral triple:

Definition 1.5.1. A **spectral triple** or **spectral data** is a triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ where \mathcal{H} is a separable Hilbert space, \mathcal{A}^∞ is a $*$ subalgebra of $\mathcal{B}(\mathcal{H})$, (not necessarily norm closed) and D is a self adjoint (typically unbounded) operator such that for all a in \mathcal{A}^∞ , the operator $[D, a]$ has a bounded extension. Such a spectral triple is also called an **odd spectral triple**. If in addition, we have γ in $\mathcal{B}(\mathcal{H})$ satisfying $\gamma = \gamma^* = \gamma^{-1}$, $D\gamma = -\gamma D$ and $[a, \gamma] = 0$ for all a in \mathcal{A}^∞ , then we say that the quadruplet $(\mathcal{A}^\infty, \mathcal{H}, D, \gamma)$ is an **even spectral triple**. The operator D is called the **Dirac operator** corresponding to the spectral triple.

Furthermore, given an abstract $*$ -algebra \mathcal{B} , an odd (even) spectral triple on \mathcal{B} is an odd (even) spectral triple $(\pi(\mathcal{B}), \mathcal{H}, D)$ (respectively, $(\pi(\mathcal{B}), \mathcal{H}, D, \gamma)$) where $\pi : \mathcal{B} \rightarrow \mathcal{B}(\mathcal{H})$ is a $*$ -homomorphism.

Since in the classical case, the Dirac operator has compact resolvent if the manifold is compact, we say that the spectral triple is of **compact type** if \mathcal{A}^∞ is unital and D has compact resolvent.

Definition 1.5.2. We say that two spectral triples $(\pi_1(\mathcal{A}), \mathcal{H}_1, D_1)$ and $(\pi_2(\mathcal{A}), \mathcal{H}_2, D_2)$ are said to be **unitarily equivalent** if there is a unitary operator $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $D_2 = UD_1U^*$ and $\pi_2(\cdot) = U\pi_1(\cdot)U^*$ where $\pi_j, j = 1, 2$ are the representations of \mathcal{A} in \mathcal{H}_j , respectively.

Next, we will give two examples of spectral triples in classical geometry and a non-classical example. We will give more examples in chapters 3 and 5.

Example 1.5.3. Let M be a smooth spin manifold. Then from Proposition 1.4.5, we see that $(C^\infty(M), \mathcal{H}, D)$ is a spectral triple over $C^\infty(M)$ and is of compact type if M is compact.

We recall that when the dimension of the manifold is even, $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$. An L^2 section s has a decomposition $s = s_1 + s_2$ where $s_1(m), s_2(m)$ belongs to $\Delta_n^+(m)$ and $\Delta_n^-(m)$ (for all m) respectively where $\Delta_n^\pm(m)$ denotes the subspace of the fibre over m . This decomposition of $L^2(S)$ induces a grading operator γ on $L^2(S)$. It can be seen that D anticommutes with γ .

Example 1.5.4. This example comes from the classical Hilbert space of forms discussed in subsection 1.5.2. One considers the self adjoint extension of the operator $d + d^*$ on $\mathcal{H} = \oplus_k \mathcal{H}^k(M)$ which is again denoted by $d + d^*$. $C^\infty(M)$ has a representation on each $\mathcal{H}^k(M)$ which gives a representation, say π on \mathcal{H} . Then it can be seen that $(C^\infty(M), \mathcal{H}, d + d^*)$ is a spectral triple and $d + d^*$ is called the **Hodge Dirac operator**. When M is compact, this spectral triple is of compact type.

Remark 1.5.5. *Let us make it clear that by a ‘classical spectral triple’ we always mean the spectral triple obtained by the Dirac operator on the spinors (so, in particular, manifolds are assumed to be Riemannian spin manifolds), and not just any spectral triple on the commutative algebra $C^\infty(M)$.*

Example 1.5.6. *The Noncommutative torus*

We recall from subsection 1.1.1 that the noncommutative 2-torus \mathcal{A}_θ is the universal C^* algebra generated by two unitaries U and V satisfying $UV = e^{2\pi i\theta}VU$ where θ is a number in $[0, 1]$.

There are two derivations d_1 and d_2 on \mathcal{A}_θ obtained by extending linearly the rule:

$$d_1(U) = U, \quad d_1(V) = 0,$$

$$d_2(U) = 0, \quad d_2(V) = V.$$

Then d_1 and d_2 are well defined on the following dense $*$ -subalgebra of \mathcal{A}_θ :

$$\mathcal{A}_\theta^\infty = \left\{ \sum_{m,n \in \mathbf{Z}} a_{mn} U^m V^n : \sup_{m,n} |m^k n^l a_{mn}| < \infty \text{ for all } k, l \text{ in } \mathbf{N} \right\}.$$

There is a unique faithful trace on \mathcal{A}_θ defined as follows:

$$\tau\left(\sum a_{mn} U^m V^n\right) = a_{00}.$$

Let $\mathcal{H} = L^2(\tau) \oplus L^2(\tau)$ where $L^2(\tau)$ denotes the GNS Hilbert space of \mathcal{A}_θ with respect to the state τ . We note that $\mathcal{A}_\theta^\infty$ is embedded as a subalgebra of $\mathcal{B}(\mathcal{H})$ by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Now, we define $D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}$.

Then, $(\mathcal{A}_\theta^\infty, \mathcal{H}, D)$ is a spectral triple of compact type. In particular, for $\theta = 0$, this coincides with the classical spectral triple on $C(\mathbb{T}^2)$.

1.5.2 The space of forms in noncommutative geometry

We start this subsection by recalling the universal space of one forms corresponding to an algebra.

Proposition 1.5.7. *Given an algebra \mathcal{B} , there is a (unique upto isomorphism) $\mathcal{B} - \mathcal{B}$ bimodule $\Omega^1(\mathcal{B})$ and a derivation $\delta : \mathcal{B} \rightarrow \Omega^1(\mathcal{B})$ (that is, $\delta(ab) = \delta(a)b + a\delta(b)$ for all a, b in \mathcal{B}), satisfying the following properties:*

(i) $\Omega^1(\mathcal{B})$ is spanned as a vector space by elements of the form $a\delta(b)$ with a, b belonging to \mathcal{B} ; and

(ii) for any $\mathcal{B} - \mathcal{B}$ bimodule E and a derivation $d : \mathcal{B} \rightarrow E$, there is an unique $\mathcal{B} - \mathcal{B}$ linear map $\eta : \Omega^1(\mathcal{B}) \rightarrow E$ such that $d = \eta \circ \delta$.

The bimodule $\Omega^1(\mathcal{B})$ is called the space of universal 1-forms on \mathcal{B} and δ is called the universal derivation.

We can also introduce universal space of higher forms on \mathcal{B} , $\Omega^k(\mathcal{B})$, say, for $k = 2, 3, \dots$, by defining them recursively as follows: $\Omega^{k+1}(\mathcal{B}) = \Omega^k(\mathcal{B}) \otimes_{\mathcal{B}} \Omega^1(\mathcal{B})$ and also set $\Omega^0(\mathcal{B}) = \mathcal{B}$.

Now we briefly discuss the notion of the noncommutative Hilbert space of forms which will need noncommutative volume form for a spectral triple of compact type. We refer to [29] (page 124 -127) and the references therein for more details.

Definition 1.5.8. A spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ of compact type is said to be Θ -summable if e^{-tD^2} is of trace class for all $t > 0$. A Θ -summable spectral triple is called finitely summable when there is some $p > 0$ such that $t^{\frac{p}{2}} \text{Tr}(e^{-tD^2})$ is bounded on $(0, \delta]$ for some $\delta > 0$. The infimum of all such p , say p' is called the dimension of the spectral triple and the spectral triple is called p' -summable.

Remark 1.5.9. We remark that the definition of Θ -summability to be used in this thesis is stronger than the one in [17] (page 390, definition 1.) in which a spectral triple is called Θ -summable if $\text{Tr}(e^{-D^2}) < \infty$.

For a Θ -summable spectral triple, let $\sigma_\lambda(T) = \frac{\text{Tr}(Te^{-\frac{1}{\lambda}D^2})}{\text{Tr}(e^{-\frac{1}{\lambda}D^2})}$ for $\lambda > 0$. We note that $\lambda \mapsto \sigma_\lambda(T)$ is bounded.

Let

$$\tau_\lambda(T) = \frac{1}{\log \lambda} \int_a^\lambda \sigma_u(T) \frac{du}{u} \text{ for } \lambda \geq a \geq e.$$

Now consider the quotient C^* algebra $\mathcal{B}_\infty = C_b([a, \infty))/C_0([a, \infty))$. Let for T in $\mathcal{B}(\mathcal{H})$, $\tau(T)$ in \mathcal{B}_∞ be the class of $\lambda \rightarrow \tau_\lambda(T)$.

For any state ω on the C^* algebra \mathcal{B}_∞ , $\text{Tr}_\omega(T) = \omega(\tau(T))$ for all T in $\mathcal{B}(\mathcal{H})$ defines a functional on $\mathcal{B}(\mathcal{H})$. As we are not going to need the choice of ω in this thesis, we will suppress the suffix ω and simply write $\text{Lim}_{t \rightarrow 0^+} \frac{\text{Tr}(Te^{-tD^2})}{\text{Tr}(e^{-tD^2})}$ for $\text{Tr}_\omega(T)$. This is a kind of Banach limit because if $\lim_{t \rightarrow 0^+} \frac{\text{Tr}(Te^{-tD^2})}{\text{Tr}(e^{-tD^2})}$ exists, then it agrees with the functional $\text{Lim}_{t \rightarrow 0^+}$. Moreover, $\text{Tr}_\omega(T)$ coincides (upto a constant) with the Dixmier trace (see chapter IV, [17]) of the operator $T|D|^{-p}$ when the spectral triple has a finite dimension $p > 0$, where $|D|^{-p}$ is to be interpreted as the inverse of the restriction of $|D|^p$ on the closure of its range. In particular, this functional gives back the volume form for the classical spectral triple on a compact Riemannian manifold.

Let $\Omega^k(\mathcal{A}^\infty)$ be the space of universal k-forms on the algebra \mathcal{A}^∞ which is spanned by $a_0\delta(a_1) \cdots \delta(a_k)$, a_i belonging to \mathcal{A}^∞ , where δ is as in Proposition 1.5.7. There

is a natural graded algebra structure on $\Omega \equiv \bigoplus_{k \geq 0} \Omega^k(\mathcal{A}^\infty)$, which also has a natural involution given by $(\delta(a))^* = -\delta(a^*)$, and using the spectral triple, we get a $*$ -representation $\Pi : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ which sends $a_0\delta(a_1) \cdots \delta(a_k)$ to $a_0d_D(a_1) \cdots d_D(a_k)$, where $d_D(a) = [D, a]$. Consider the state τ on $\mathcal{B}(\mathcal{H})$ given by, $\tau(X) = \lim_{t \rightarrow 0^+} \frac{\text{Tr}(Xe^{-tD^2})}{\text{Tr}(e^{-tD^2})}$, where \lim is as above. Using τ , we define a positive semi definite sesquilinear form on $\Omega^k(\mathcal{A}^\infty)$ by setting $\langle w, \eta \rangle = \tau(\Pi(w)^* \Pi(\eta))$. Let $K^k = \{w \in \Omega^k(\mathcal{A}^\infty) : \langle w, w \rangle = 0\}$, for $k \geq 0$, and $K^{-1} := (0)$. Let $\overline{\Omega_D^k}$ be the Hilbert space obtained by completing the quotient $\Omega^k(\mathcal{A}^\infty)/K^k$ with respect to the inner product mentioned above, and we define $\mathcal{H}_D^k := P_k^\perp \overline{\Omega_D^k}$, where P_k denotes the projection onto the closed subspace generated by $\delta(K^{k-1})$. The map $D' := d + d^* \equiv d_D + d_D^*$ on $\mathcal{H}_{d+d^*} := \bigoplus_{k \geq 0} \mathcal{H}_D^k$ has a self-adjoint extension (which is again denoted by $d + d^*$). Clearly, \mathcal{H}_D^k has a total set consisting of elements of the form $[a_0\delta(a_1) \cdots \delta(a_k)]$, with a_i in \mathcal{A}^∞ and where $[\omega]$ denotes the equivalence class $P_k^\perp(w + K^k)$ for ω belonging to $\Omega^k(\mathcal{A}^\infty)$. There is a $*$ -representation $\pi_{d+d^*} : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_{d+d^*})$, given by $\pi_{d+d^*}(a)([a_0\delta(a_1) \cdots \delta(a_k)]) = [aa_0\delta(a_1) \cdots \delta(a_k)]$. Then it is easy to see that

Proposition 1.5.10. $(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d + d^*)$ is a spectral triple.

1.5.3 Laplacian in Noncommutative geometry

Now, we discuss the notion of Laplacian in noncommutative geometry as introduced in [30]. Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple of compact type. To define the Laplacian in the noncommutative case (as in [30]), we need the following assumptions on the spectral triple.

Assumptions

1. $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a compact type spectral triple.
 2. It is QC^∞ , that is, \mathcal{A}^∞ and $\{[D, a], a \in \mathcal{A}^\infty\}$ are contained in the domains of all powers of the derivation $[[D], \cdot]$.
 3. The unbounded densely defined map d_D from \mathcal{H}_D^0 to \mathcal{H}_D^1 given by $d_D(a) = [D, a]$ for a in \mathcal{A}^∞ , is closable.
 4. $\mathcal{L} := -d_D^* d_D$ has \mathcal{A}^∞ in its domain, and it is left invariant by \mathcal{L} .
- Under assumption 2., τ defined by $\tau(X) = \lim_{t \rightarrow 0^+} \frac{\text{Tr}(Xe^{-tD^2})}{\text{Tr}(e^{-tD^2})}$ is a positive trace on the C^* -subalgebra generated by \mathcal{A}^∞ and $\{[D, a] : a \in \mathcal{A}^\infty\}$.
5. We assume that it is also faithful on this subalgebra.

Then, $\mathcal{L} = -d_D^* d_D$ is defined to be the Laplacian for the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$. It coincides with the Hodge Laplacian $-d^* d$ (restricted on space of smooth functions) in the classical case, where d denotes the de-Rham differential.

The linear span of eigenvectors of \mathcal{L} , which is a subspace of \mathcal{A}^∞ , is denoted by \mathcal{A}_0^∞ , and the $*$ -subalgebra of \mathcal{A}^∞ generated by \mathcal{A}_0^∞ is denoted by \mathcal{A}_0 .

Chapter 2

Quantum isometry groups: approach based on Laplacian

The idea of quantum isometry group of a noncommutative manifold (given by a spectral triple), which has been defined by Goswami, is motivated by the definition and study of quantum permutation groups of finite sets and finite graphs by a number of mathematicians (see, e.g. [1], [2], [60], [61] and references therein).

In this chapter, we first recall the definition of quantum isometry groups as proposed in [30] and then compute it for some examples.

2.1 Formulation of the quantum isometry group

2.1.1 Characterization of isometry group for a compact Riemannian manifold

Let M be a compact Riemannian manifold. Consider the category with objects being the pairs (G, α) where G is a compact metrizable group acting on M by the smooth and isometric action α . If (G_1, α) and (G_2, β) are two objects in this category, $\text{Mor}((G_1, \alpha), (G_2, \beta))$ consists of group homomorphisms π from G_1 to G_2 such that $\beta \circ \pi = \alpha$. Then the isometry group of M is the universal object in this category.

More generally, the isometry group of a classical compact Riemannian manifold, viewed as a compact metrizable space (forgetting the group structure), can be seen to be the universal object of a category whose object class consists of subsets (not generally subgroups) of the set of smooth isometries of the manifold. Then it can be proved that this universal compact set has a canonical group structure. Thus, motivated by the ideas of Woronowicz and Soltan ([53], [68]), Goswami considered in [30] a bigger category with objects as the pair (S, f) where S is a compact metrizable space and $f : S \times M \rightarrow M$ such that the map from M to itself defined by $m \mapsto f(s, m)$ is a

smooth isometry for all s in S . The morphism set is defined as above (replacing group homomorphisms by continuous set maps).

Therefore, to define the quantum isometry group, it is reasonable to consider a category of compact quantum groups which act on the manifold (or more generally, on a noncommutative manifold given by spectral triple) in a ‘nice’ way, preserving the Riemannian structure in some suitable sense, which is precisely formulated in [30], where it is also proven that a universal object in the category of such quantum groups does exist if one makes some natural regularity assumptions on the spectral triple.

2.1.2 The definition and existence of the quantum isometry group

Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a Θ -summable spectral triple of compact type. We recall from section 1.5 the Hilbert spaces of k -forms $\mathcal{H}_D^k, k = 0, 1, 2, \dots$ and also the Laplacian $\mathcal{L} = -d_D^* d_D$.

To define the quantum isometry group, we need the following assumptions:

Assumptions

1. d_D is closable and $\mathcal{A}^\infty \subseteq \text{Dom}(\mathcal{L})$ where \mathcal{A}^∞ is viewed as a dense subspace of \mathcal{H}_D^0 .
2. \mathcal{L} has compact resolvents.
3. $\mathcal{L}(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$.
4. Each eigenvector of \mathcal{L} (which has a discrete spectrum, hence a complete set of eigenvectors) belongs to \mathcal{A}^∞ .
5. (connectedness assumption) The kernel of \mathcal{L} is one dimensional, spanned by the identity 1 of \mathcal{A}^∞ , viewed as a unit vector in \mathcal{H}_D^0 .
6. The complex linear span of the eigenvectors of \mathcal{L} , denoted by \mathcal{A}_0^∞ is norm dense in \mathcal{A}^∞ .

Definition 2.1.1. *We say that a spectral triple satisfying the assumptions 1. - 6. is admissible.*

The following result is contained in Remark 2.16 of [30].

Proposition 2.1.2. *If an admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ satisfies the condition $\bigcap \text{Dom}(\mathcal{L}^n) = \mathcal{A}^\infty$, and if $\alpha : \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}} \otimes S$ is a smooth isometric action on \mathcal{A}^∞ by a CQG S , then for all state ϕ on S , $\alpha_\phi (= (\text{id} \otimes \phi)\alpha)$ keeps \mathcal{A}^∞ invariant.*

In view of the characterization of smooth isometric action on a classical compact manifold (Proposition 1.4.1 and Proposition 1.4.2 in Chapter 1), Goswami gave the following definition in [30].

Definition 2.1.3. *A quantum family of smooth isometries of the noncommutative manifold \mathcal{A}^∞ (or more precisely on the corresponding spectral triple) is a pair (S, α) where*

\mathcal{S} is a separable unital C^* algebra, $\alpha : \overline{\mathcal{A}} \rightarrow \overline{\mathcal{A}} \otimes \mathcal{S}$ (where $\overline{\mathcal{A}}$ denotes the C^* algebra obtained by completing \mathcal{A}^∞ in the norm of $\mathcal{B}(\mathcal{H}_D^0)$) is a unital C^* homomorphism, satisfying the following:

- a. $\overline{\text{Sp}}(\alpha(\overline{\mathcal{A}}))(1 \otimes \mathcal{S}) = \overline{\mathcal{A}} \otimes \mathcal{S}$
- b. $\alpha_\phi = (\text{id} \otimes \phi)\alpha$ maps \mathcal{A}_0^∞ into itself and commutes with \mathcal{L} on \mathcal{A}_0^∞ , for every state ϕ on \mathcal{S} .

In case, the C^* algebra has a coproduct Δ such that (\mathcal{S}, Δ) is a compact quantum group and α is an action of (\mathcal{S}, Δ) on $\overline{\mathcal{A}}$, we say that (\mathcal{S}, Δ) acts smoothly and isometrically on the noncommutative manifold.

Notations

1. We will denote by $\mathbf{Q}^\mathcal{L}$ the category with the object class consisting of all quantum families of isometries (\mathcal{S}, α) of the given noncommutative manifold, and the set of morphisms $\text{Mor}((\mathcal{S}, \alpha), (\mathcal{S}', \alpha'))$ being the set of unital C^* homomorphisms $\phi : \mathcal{S} \rightarrow \mathcal{S}'$ satisfying $(\text{id} \otimes \phi)\alpha = \alpha'$.

2. We will denote by $\mathbf{Q}'_\mathcal{L}$ the category whose objects are triplets $(\mathcal{S}, \Delta, \alpha)$ where (\mathcal{S}, Δ) is a CQG acting smoothly and isometrically on the given noncommutative manifold, with α being the corresponding action. The morphisms are the homomorphisms of compact quantum groups which are also morphisms of the underlying quantum families.

Let $\{\lambda_1, \lambda_2, \dots\}$ be the set of eigenvalues of \mathcal{L} , with V_i being the corresponding (finite dimensional) eigenspace. We will denote by \mathcal{U}_i the Wang algebra $A_{u, d_i}(I)$ (as introduced in the chapter 1) where d_i is the dimension of the subspace V_i . We fix a representation $\beta_i : V_i \rightarrow V_i \otimes \mathcal{U}_i$ on the Hilbert space V_i , given by $\beta_i(e_{ij}) = \sum_k e_{ik} \otimes u_{kj}^{(i)}$ for $j = 1, 2, \dots, d_i$, where $\{e_{ij}\}$ is an orthonormal basis for V_i , and $u^{(i)} \equiv u_{kj}^{(i)}$ are the generators of \mathcal{U}_i . Thus, both $u^{(i)}$ and $\overline{u^{(i)}}$ are unitaries. The representations β_i canonically induce the free product representation $\beta = *_i \beta_i$ of the free product CQG $\mathcal{U} = *_i \mathcal{U}_i$ on the Hilbert space \mathcal{H}_D^0 such that the restriction of β on V_i coincides with β_i for all i .

The following Lemma (Lemma 2.12 of [30]) will be needed later and hence we record it.

Lemma 2.1.4. *Consider an admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ and let (\mathcal{S}, α) be a quantum family of smooth isometries of the spectral triple. Moreover, assume that the action is faithful in the sense that there is no proper C^* subalgebra \mathcal{S}_1 of \mathcal{S} such that $\alpha(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty \otimes \mathcal{S}_1$. Then $\tilde{\alpha} : \mathcal{A}^\infty \otimes \mathcal{S} \rightarrow \mathcal{A}^\infty \otimes \mathcal{S}$ defined by $\tilde{\alpha}(a \otimes b) = \alpha(a)(1 \otimes b)$ extends to an \mathcal{S} linear unitary on the Hilbert \mathcal{S} module $\mathcal{H}_D^0 \otimes \mathcal{S}$, denoted again by $\tilde{\alpha}$. Moreover, we can find a C^* isomorphism $\phi : \mathcal{U}/\mathcal{I} \rightarrow \mathcal{S}$ between \mathcal{S} and a quotient of \mathcal{U} by a C^* ideal \mathcal{I} of \mathcal{U} , such that $\alpha = (\text{id} \otimes \phi) \circ (\text{id} \otimes \Pi_\mathcal{I}) \circ \beta$ on $\mathcal{A}^\infty \subseteq \mathcal{H}_D^0$, where $\Pi_\mathcal{I}$ denotes the quotient map from \mathcal{U} to \mathcal{U}/\mathcal{I} .*

If furthermore, there is a CQG structure on \mathcal{S} given by a coproduct Δ such that α is a C^* action of a CQG on $\overline{\mathcal{A}}$, then the map $\alpha : \mathcal{A}^\infty \rightarrow \mathcal{A}^\infty \otimes \mathcal{S}$ extends to a unitary representation (denoted again by α) of the CQG (\mathcal{S}, Δ) on \mathcal{H}_D^0 . In this case, the ideal \mathcal{I} is a Woronowicz C^* ideal and the C^* isomorphism $\phi : \mathcal{U}/\mathcal{I} \rightarrow \mathcal{S}$ is a morphism of CQG's.

Using this, the following result has been proved in [30], which defines and gives the existence of $QISO^\mathcal{L}$.

Theorem 2.1.5. *For any admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, the category $\mathbf{Q}^\mathcal{L}$ has a universal object denoted by $(QISO^\mathcal{L}, \alpha_0)$. Moreover, $QISO^\mathcal{L}$ has a coproduct Δ_0 such that $(QISO^\mathcal{L}, \Delta_0)$ is a CQG and $(QISO^\mathcal{L}, \Delta_0, \alpha_0)$ is a universal object in the category $\mathbf{Q}'_\mathcal{L}$. The action α_0 is faithful.*

We very briefly outline the main ideas of the proof. The universal object $QISO^\mathcal{L}$ is constructed as a suitable quotient of \mathcal{U} . Let \mathcal{F} be the collection of all those C^* -ideals \mathcal{I} of \mathcal{U} such that the composition $\Gamma_\mathcal{I} = (\text{id} \otimes \Pi_\mathcal{I}) \circ \beta : \mathcal{A}_0^\infty \rightarrow \mathcal{A}_0^\infty \otimes_{\text{alg}} (\mathcal{U}/\mathcal{I})$ extends to a C^* -homomorphism from $\overline{\mathcal{A}}$ to $\overline{\mathcal{A}} \otimes (\mathcal{U}/\mathcal{I})$. Then it can be shown that $\mathcal{I}_0 (= \cap_{\mathcal{I} \in \mathcal{F}} \mathcal{I})$ is again a member of \mathcal{F} and $(\mathcal{U}/\mathcal{I}_0, \Gamma_{\mathcal{I}_0})$ is the required universal object. Thus,

Remark 2.1.6. $QISO^\mathcal{L}$ is a quantum subgroup of the CQG $\mathcal{U} = *_i A_{u, d_i}(I)$. As $A_{u, d_i}(I)$ satisfies $\kappa^2 = \text{id}$, (by Remark 1.2.29) the same is satisfied by $QISO^\mathcal{L}$ so that by Remark 1.2.20, $QISO^\mathcal{L}$ has tracial Haar state.

Remark 2.1.7. It is proved in [30] that to ensure the existence of $QISO^\mathcal{L}$, the assumption (5) can be replaced by the condition that the action α is τ preserving, that is, $(\tau \otimes \text{id})\alpha(a) = \tau(a)$. In [30] it was also shown (Lemma 2.5, $b \Rightarrow a$) that for an isometric group action on a not necessarily connected classical manifold, the volume functional is automatically preserved. It can be easily seen that the proof goes verbatim for a quantum group action, and consequently we get the existence of $QISO^\mathcal{L}$ for a (not necessarily connected) compact Riemannian manifold.

Unitary representation of $QISO^\mathcal{L}$ on a spectral triple

We shall also need the following result proved in section 2.4 of [30].

Proposition 2.1.8. $QISO^\mathcal{L}$ has a unitary representation $U \equiv U_\mathcal{L}$ on \mathcal{H}_D such that U commutes with $d + d^*$. Let δ be as in subsection 1.5.2. On the Hilbert space of k -forms, that is. \mathcal{H}_D^k , U is defined by:

$$U([a_0 \delta(a_1) \cdots \delta(a_k)] \otimes q) = [a_0^{(1)} \delta(a_1^{(1)}) \cdots \delta(a_k^{(1)})] \otimes (a_0^{(2)} a_1^{(2)} \cdots a_k^{(2)}) q,$$

where q belongs to $QISO^\mathcal{L}$, a_i belongs to \mathcal{A}_0^∞ , and for x in \mathcal{A}_0 , (the $*$ -subalgebra generated by the eigenvectors of \mathcal{L}) we write in Sweedler notation $\alpha(x) = x^{(1)} \otimes x^{(2)} \in \mathcal{A}_0 \otimes (QISO^\mathcal{L})_0$ (α denotes the action of $QISO^\mathcal{L}$).

2.2 Computation of $QISO^\mathcal{L}$

Here we compute $QISO^\mathcal{L}$ for three commutative examples, viz: the sphere, the circle and the n tori. In Chapter 4, we will be able to compute it for two noncommutative examples, namely \mathcal{A}_θ and S_θ^n by using Theorem 4.4.7.

2.2.1 The commutative spheres

Let $QISO^\mathcal{L}$ be the quantum isometry group of S^2 and let α be the action of $QISO^\mathcal{L}$ on $C(S^2)$. Let \mathcal{L} be the Laplacian on S^2 defined as

$$\mathcal{L} = \frac{\partial^2}{\partial \theta^2} + \cot(\theta) \frac{\partial}{\partial \theta} + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \psi^2},$$

where the cartesian coordinates x_1, x_2, x_3 for S^2 are given by $x_1 = r \cos \psi \sin \theta$, $x_2 = r \sin \psi \sin \theta$, $x_3 = r \cos \theta$. In the cartesian coordinates, $\mathcal{L} = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$.

The eigenspaces of \mathcal{L} on S^2 are of the form

$$E_k = \text{Sp}\{(c_1 X_1 + c_2 X_2 + c_3 X_3)^k : c_i \in \mathbb{C}, i = 1, 2, 3, \sum c_i^2 = 0\},$$

where $k \geq 1$. E_k consists of harmonic homogeneous polynomials of degree k on \mathbb{R}^3 restricted to S^2 (See [33], page 29-30).

We begin with the following lemma, which says that any smooth isometric action by a quantum group must be ‘linear’.

Lemma 2.2.1. *The action α satisfies $\alpha(x_i) = \sum_{j=1}^3 x_j \otimes Q_{ij}$ where Q_{ij} belongs to $QISO^\mathcal{L}$, $i = 1, 2, 3$.*

Proof : Since α is a smooth isometric action of $QISO^\mathcal{L}$ on $C(S^2)$, α has to preserve the eigenspaces of the Laplacian \mathcal{L} . In particular, it has to preserve $E_1 = \text{Sp}\{c_1 x_1 + c_2 x_2 + c_3 x_3 : c_i \in \mathbb{C}, i = 1, 2, 3, \sum_{i=1}^3 c_i^2 = 0\}$.

Now note that $x_1 + ix_2$, $x_1 - ix_2$ are in E_1 , hence x_1, x_2 are in E_1 . Similarly x_3 belongs to E_1 too. Therefore $E_1 = \text{Sp}\{x_1, x_2, x_3\}$, which completes the proof of the lemma. \square

Now, we state and prove the main result of this section, which identifies $QISO^\mathcal{L}$ with the commutative C^* algebra of continuous functions on the isometry group of S^2 , that is $O(3)$.

Theorem 2.2.2. *The quantum isometry group $QISO^{\mathcal{L}}$ is commutative as a C^* algebra, and hence $QISO^{\mathcal{L}} \cong C(O(3))$.*

Proof : We begin with the expression

$$\alpha(x_i) = \sum_{j=1}^3 x_j \otimes Q_{ij}, \quad i = 1, 2, 3,$$

and also note that x_1, x_2, x_3 form a basis of E_1 and $\{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3\}$ is a basis of E_2 . Since $x_i^* = x_i$ for each i and α is a $*$ -homomorphism, we must have $Q_{ij}^* = Q_{ij}$ for all $i, j = 1, 2, 3$. Moreover, the condition $x_1^2 + x_2^2 + x_3^2 = 1$ and the fact that α is a homomorphism gives:

$$Q_{1j}^2 + Q_{2j}^2 + Q_{3j}^2 = 1, \quad \forall j = 1, 2, 3.$$

Again, the condition that x_i, x_j commute for all i, j gives

$$Q_{ij}Q_{kj} = Q_{kj}Q_{ij} \quad \forall i, j, k, \quad (2.2.1)$$

$$Q_{ik}Q_{jl} + Q_{il}Q_{jk} = Q_{jk}Q_{il} + Q_{jl}Q_{ik}. \quad (2.2.2)$$

Now, it follows from the Lemma 2.1.4 that $\tilde{\alpha} : C(S^2) \otimes QISO^{\mathcal{L}} \rightarrow C(S^2) \otimes QISO^{\mathcal{L}}$ defined by $\tilde{\alpha}(X \otimes Y) = \alpha(X)(1 \otimes Y)$ extends to a unitary of the Hilbert $QISO^{\mathcal{L}}$ -module $L^2(S^2) \otimes QISO^{\mathcal{L}}$ (or in other words, α extends to a unitary representation of $QISO^{\mathcal{L}}$ on $L^2(S^2)$). But α keeps $V = \text{Sp}\{x_1, x_2, x_3\}$ invariant. So α is a unitary representation of $QISO^{\mathcal{L}}$ on V , that is $Q = ((Q_{ij}))$ belonging to $M_3(QISO^{\mathcal{L}})$ is a unitary, hence $Q^{-1} = Q^* = Q^T$, since in this case entries of Q are self-adjoint elements.

Clearly, the matrix Q is a 3-dimensional unitary representation of $QISO^{\mathcal{L}}$. From (4) of Proposition 1.2.23, the antipode κ on the matrix elements of a finite-dimensional unitary representation $U^\alpha \equiv (u_{pq}^\alpha)$ is given by $\kappa(u_{pq}^\alpha) = (u_{qp}^\alpha)^*$.

So we obtain

$$\kappa(Q_{ij}) = Q_{ij}^{-1} = Q_{ij}^T = Q_{ji}. \quad (2.2.3)$$

Now from (2.2.1), we have $Q_{ij}Q_{kj} = Q_{kj}Q_{ij}$. Applying κ on this equation and using the fact that κ is an antihomomorphism along with (2.2.3), we have $Q_{jk}Q_{ji} = Q_{ji}Q_{jk}$. Similarly, applying κ on (2.2.2), we get

$$Q_{lj}Q_{ki} + Q_{kj}Q_{li} = Q_{li}Q_{kj} + Q_{ki}Q_{lj} \quad \forall i, j, k, l.$$

Interchanging between k and i and also between l, j gives

$$Q_{jl}Q_{ik} + Q_{il}Q_{jk} = Q_{jk}Q_{il} + Q_{ik}Q_{jl} \quad \forall i, j, k, l. \quad (2.2.4)$$

Now, by (2.2.2) - (2.2.4), we have

$$[Q_{ik}, Q_{jl}] = [Q_{jl}, Q_{ik}],$$

hence

$$[Q_{ik}, Q_{jl}] = 0.$$

Therefore the entries of the matrix Q commute among themselves. However, by faithfulness of the action of $QISO^\mathcal{L}$, it is clear that the C^* -subalgebra generated by entries of Q (which forms a quantum subgroup of $QISO^\mathcal{L}$ acting on $C(S^2)$ isometrically) must be the same as $QISO^\mathcal{L}$, so $QISO^\mathcal{L}$ is commutative.

So $QISO^\mathcal{L} = C(G)$ for some compact group G acting by isometry on $C(S^2)$ and G is clearly universal in the category of compact metrizable groups acting on S^2 isometrically, so that $G \cong O(3)$. \square

Remark 2.2.3. *Similarly, it can be shown that $QISO(S^n)$ is commutative for all $n \geq 2$.*

2.2.2 The commutative one-torus

Let $\mathcal{C} = C(S^1)$ be the C^* -algebra of continuous functions on the one-torus S^1 . Let us denote by z and \bar{z} the identity function (which is the generator of $C(S^1)$) and its conjugate respectively. The Laplacian coming from the standard Riemannian metric is given by $\mathcal{L}(z^n) = -n^2 z^n$, for n in \mathbf{Z} , hence the eigenspace corresponding to the eigenvalue -1 is spanned by z and \bar{z} . Thus, the action of a compact quantum group acting smoothly and isometrically (and faithfully) on $C(S^1)$ must be *linear* in the sense that its action must map z into an element of the form $z \otimes A + \bar{z} \otimes B$. However, we show below that this forces the quantum group to be commutative as a C^* algebra, that is it must be the function algebra of some compact group.

Theorem 2.2.4. *Let α be a faithful, smooth and linear action of a compact quantum group (\mathcal{Q}, Δ) on $C(S^1)$ defined by $\alpha(z) = z \otimes A + \bar{z} \otimes B$. Then \mathcal{Q} is a commutative C^* algebra.*

Proof : By the assumption of faithfulness, it is clear that \mathcal{Q} is generated (as a unital C^* algebra) by A and B . Moreover, recall that smoothness in particular means that A and B must belong to the algebra \mathcal{Q}_0 spanned by matrix elements of irreducible representations of \mathcal{Q} . Since $z\bar{z} = \bar{z}z = 1$ and α is a $*$ -homomorphism, we have $\alpha(z)\alpha(\bar{z}) = \alpha(\bar{z})\alpha(z) = 1 \otimes 1$.

Comparing coefficients of z^2, \bar{z}^2 and 1 in both hand sides of the relation $\alpha(z)\alpha(\bar{z}) = 1 \otimes 1$, we get

$$AB^* = BA^* = 0, \quad AA^* + BB^* = 1. \quad (2.2.5)$$

Similarly, $\alpha(\bar{z})\alpha(z) = 1 \otimes 1$ gives

$$B^*A = A^*B = 0, \quad A^*A + B^*B = 1. \quad (2.2.6)$$

Let $U = A + B$, $P = A^*A$, $Q = AA^*$. Then it follows from (2.2.5) and (2.2.6) that U is a unitary and P is a projection since P is self adjoint and

$$P^2 = A^*AA^*A = A^*A(1 - B^*B) = A^*A - A^*AB^*B = A^*A = P.$$

Moreover ,

$$\begin{aligned} UP &= (A + B)A^*A = AA^*A + BA^*A = AA^*A \\ &\quad (\text{since } BA^* = 0 \text{ from (2.2.5)}) \\ &= A(1 - B^*B) = A - AB^*B = A. \end{aligned}$$

Thus, $A = UP$, $B = U - UP = U(1 - P) \equiv UP^\perp$, so $\mathcal{Q} = C^*(A, B) = C^*(U, P)$.

We can rewrite the action α as follows:

$$\alpha(z) = z \otimes UP + \bar{z} \otimes UP^\perp.$$

The coproduct Δ can easily be calculated from the requirement $(\text{id} \otimes \Delta)\alpha = (\alpha \otimes \text{id})\alpha$, and it is given by :

$$\Delta(UP) = UP \otimes UP + P^\perp U^{-1} \otimes UP^\perp, \quad (2.2.7)$$

$$\Delta(UP^\perp) = UP^\perp \otimes UP + PU^{-1} \otimes UP^\perp. \quad (2.2.8)$$

From this, we get

$$\Delta(U) = U \otimes UP + U^{-1} \otimes UP^\perp, \quad (2.2.9)$$

$$\Delta(P) = \Delta(U^{-1})\Delta(UP) = P \otimes P + UP^\perp U^{-1} \otimes P^\perp. \quad (2.2.10)$$

It can be checked that Δ given by the above expression is coassociative.

Let h denote the right-invariant Haar state on \mathcal{Q} . By the general theory of compact quantum groups, h must be faithful on \mathcal{Q}_0 . We have (by right-invariance of h):

$$(\text{id} \otimes h)(P \otimes P + UP^\perp U^{-1} \otimes P^\perp) = h(P)1.$$

That is, we have

$$h(P^\perp)UP^\perp U^{-1} = h(P)P^\perp. \quad (2.2.11)$$

Since P is a positive element in \mathcal{Q}_0 and h is faithful on \mathcal{Q}_0 , $h(P) = 0$ if and only if $P = 0$. Similarly, $h(P^\perp) = 0$, that is $h(P) = 1$, if and only if $P = 1$. However, if P is either 0 or 1, clearly $\mathcal{Q} = C^*(U, P) = C^*(U)$, which is commutative. On the other hand, if we assume that P is not a trivial projection, then $h(P)$ is strictly between 0 and 1, and we have from (2.2.11)

$$UP^\perp U^{-1} = \frac{h(P)}{1 - h(P)} P^\perp.$$

Since both $UP^\perp U^{-1}$ and P^\perp are nontrivial projections, they can be scalar multiples of each other if and only if they are equal, so we conclude that $UP^\perp U^{-1} = P^\perp$, that is U commutes with P^\perp , hence with P , and \mathcal{Q} is commutative. \square

2.2.3 The commutative n-tori

Consider $C(\mathbb{T}^n)$ as the universal commutative C^* algebra generated by n commuting unitaries U_1, U_2, \dots, U_n . It is clear that the set $\{U_i^m U_j^n : m, n \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(C(\mathbb{T}^n), \tau_0)$, where τ_0 denotes the unique faithful normalized trace on $C(\mathbb{T}^n)$ given by, $\tau_0(\sum a_{mn} U_i^m U_j^n) = a_{00}$ which is just the integration against the Haar measure. We shall denote by $\langle A, B \rangle = \tau_0(A^* B)$ the inner product on $\mathcal{H}_0 := L^2(C(\mathbb{T}^n), \tau_0)$. Let $C(\mathbb{T}^n)^{\text{fin}}$ be the unital $*$ -subalgebra generated by finite complex linear combinations of $U^m V^n$, $m, n \in \mathbf{Z}$. The Laplacian \mathcal{L} is given by $\mathcal{L}(U_1^{m_1} \dots U_n^{m_n}) = -(m_1^2 + \dots m_n^2) U_1^{m_1} \dots U_n^{m_n}$, and it is also easy to see that the algebraic span of eigenvectors of \mathcal{L} is nothing but the space $C(\mathbb{T}^n)^{\text{fin}}$, and moreover, all the assumptions **1.** - **6.** in subsection 2.1.2 required for defining the quantum isometry group are satisfied.

Let $QISO^{\mathcal{L}}$ be the quantum isometry group coming from the above Laplacian, with the smooth isometric action of $QISO^{\mathcal{L}}$ on $C(\mathbb{T}^n)$ given by $\alpha : C(\mathbb{T}^n) \rightarrow C(\mathbb{T}^n) \otimes QISO^{\mathcal{L}}$. By definition, α must keep invariant the eigenspace of \mathcal{L} corresponding to the eigenvalue -1 , spanned by $U_1, \dots, U_n, U_1^{-1}, \dots, U_n^{-1}$. Thus, the action α is given by:

$$\alpha(U_i) = \sum_{j=1}^n U_j \otimes A_{ij} + \sum_{j=1}^n U_j^{-1} \otimes B_{ij},$$

where A_{ij}, B_{ij} are in $QISO^{\mathcal{L}}$, $i, j = 1, 2, \dots, n$. By faithfulness of the action of quantum isometry group, the norm-closure of the unital $*$ -algebra generated by $\{A_{ij}, B_{ij} : i, j = 1, 2, \dots, n\}$ must be the whole of $QISO^{\mathcal{L}}$.

Next we derive a number of conditions on $A_{ij}, B_{ij}, i, j = 1, 2, \dots, n$ using the fact that α is a $*$ homomorphism.

Lemma 2.2.5. *The condition $U^*U = 1 = UU^*$ gives:*

$$\sum_j (A_{ij}^* A_{ij} + B_{ij}^* B_{ij}) = 1, \quad (2.2.12)$$

$$B_{ij}^* A_{ik} + B_{ik}^* A_{ij} = 0 \quad \forall j \neq k, \quad (2.2.13)$$

$$A_{ij}^* B_{ik} + A_{ik}^* B_{ij} = 0 \quad \forall j \neq k, \quad (2.2.14)$$

$$A_{ij}^* B_{ij} = B_{ij}^* A_{ij} = 0, \quad (2.2.15)$$

$$\sum_j (A_{ij} A_{ij}^* + B_{ij} B_{ij}^*) = 1, \quad (2.2.16)$$

$$A_{ik} B_{ij}^* + A_{ij} B_{ik}^* = 0 \quad \forall j \neq k, \quad (2.2.17)$$

$$B_{ik} A_{ij}^* + B_{ij} A_{ik}^* = 0 \quad \forall j \neq k, \quad (2.2.18)$$

$$A_{ij} B_{ij}^* = B_{ij} A_{ij}^* = 0. \quad (2.2.19)$$

Proof : We get (2.2.12) - (2.2.15) by using the condition $U_i^* U_i = 1$ along with the fact that α is a homomorphism and then comparing the coefficients of $1, U_j U_k, U_j^{-1} U_k^{-1}$, (for $j \neq k$), U_j^{-2}, U_k^{-2} .

Similarly the condition $U_i U_i^* = 1$ gives (2.2.16) - (2.2.19). \square

Now, for all $i \neq j$, $U_i^* U_j, U_i U_j^*$ and $U_i U_j$ belong to the eigenspace of the Laplacian with eigenvalue -2 , while U_k^2, U_k^{-2} belong to the eigenspace corresponding to the eigenvalue -4 . As α preserves the eigenspaces of the Laplacian, the coefficients of U_k^2, U_k^{-2} are zero for all k in $\alpha(U_i^* U_j), \alpha(U_i U_j^*), \alpha(U_i U_j)$ for all $i \neq j$.

We use this observation in the next lemma.

Lemma 2.2.6. *For all k and for all $i \neq j$,*

$$B_{ik}^* A_{jk} = A_{ik}^* B_{jk} = 0, \quad (2.2.20)$$

$$A_{ik} B_{jk} = B_{ik} A_{jk}^* = 0, \quad (2.2.21)$$

$$A_{ik} A_{jk} = B_{ik} B_{jk} = 0. \quad (2.2.22)$$

Proof : The equation (2.2.20) is obtained from the coefficients of U_k^2 and U_k^{-2} in $\alpha(U_i^* U_j)$ while (2.2.21) and (2.2.22) are obtained from the same coefficients in $\alpha(U_i U_j^*)$ and $\alpha(U_i U_j)$ respectively. \square

Now, by Lemma 2.1.4 it follows that $\tilde{\alpha} : C(\mathbb{T}^n) \otimes QISO^{\mathcal{L}} \rightarrow C(\mathbb{T}^n) \otimes QISO^{\mathcal{L}}$ defined by $\tilde{\alpha}(X \otimes Y) = \alpha(X)(1 \otimes Y)$ extends to a unitary of the Hilbert $QISO^{\mathcal{L}}$ -module $L^2(C(\mathbb{T}^n), \tau) \otimes QISO^{\mathcal{L}}$ (or in other words, α extends to a unitary representation of

$QISO^\mathcal{L}$ on $L^2(C(\mathbb{T}^n), \tau)$). But α keeps $W = \text{Sp}\{U_i, U_i^* : 1 \leq i \leq n\}$ invariant. So α is a unitary representation of $QISO^\mathcal{L}$ on W . Hence, the matrix (say M) corresponding to the $2n$ dimensional representation of $QISO^\mathcal{L}$ on W is a unitary in $M_{2n}(QISO^\mathcal{L})$.

From the definition of the action it follows that $M = \begin{pmatrix} A_{ij} & B_{ij}^* \\ B_{ij} & A_{ij}^* \end{pmatrix}$.

Since M is the matrix corresponding to a finite dimensional unitary representation, using (4) of Proposition 1.2.23, we have $(k(M_{kl})) = \begin{pmatrix} A_{ji}^* & B_{ji}^* \\ B_{ji} & A_{ji} \end{pmatrix}$ (κ is the antipode of $QISO^\mathcal{L}$).

Now we apply the antipode κ to get some more relations.

Lemma 2.2.7. :

For all k and $i \neq j$,

$$A_{kj}^* A_{ki}^* = B_{kj} B_{ki} = A_{kj}^* B_{ki}^* = B_{kj} A_{ki} = B_{kj} A_{ki}^* = A_{kj} B_{ki} = 0. \quad (2.2.23)$$

Proof : The result follows by applying κ on the equations $A_{ik} A_{jk} = B_{ik} B_{jk} = B_{ik}^* A_{jk} = A_{ik}^* B_{jk} = A_{ik} B_{jk} = B_{ik} A_{jk}^* = 0$ obtained from Lemma 2.2.6. \square

Lemma 2.2.8. :

A_{li} is a normal partial isometry for all l, i and hence has same domain and range.

Proof : From the relation (2.2.12) in Lemma 2.2.5, we have by applying κ , $\sum (A_{ji}^* A_{ji} + B_{ji} B_{ji}^*) = 1$. Applying A_{li} on the right of this equation, we have

$$A_{li}^* A_{li} A_{li} + \sum_{j \neq l} (A_{ji}^* A_{ji} A_{li} + B_{li} B_{li}^* A_{li}) + \sum_{j \neq l} B_{ji} B_{ji}^* A_{li} = A_{li}.$$

From Lemma 2.2.6, we have $A_{ji} A_{li} = 0$ and $B_{ji}^* A_{li} = 0$ for all $j \neq l$. Moreover, from Lemma 2.2.5, we have $B_{li}^* A_{li} = 0$. Applying these to the above equation, we have

$$A_{li}^* A_{li} A_{li} = A_{li}. \quad (2.2.24)$$

Again, from the relation $\sum_j (A_{ij} A_{ij}^* + B_{ij} B_{ij}^*) = 1$ for all i in Lemma 2.2.5, applying κ and multiplying by A_{li}^* on the right, we have $A_{li} A_{li}^* A_{li}^* + \sum_{j \neq l} A_{ji} A_{ji}^* A_{li}^* + B_{li}^* B_{li} A_{li}^* + \sum_{j \neq l} B_{ji}^* B_{ji} A_{li}^* = A_{li}^*$. From Lemma 2.2.6, we have $A_{li} A_{ji} = 0$ for all $j \neq l$ (hence $A_{ji}^* A_{li}^* = 0$) and $B_{ji} A_{li}^* = 0$. Moreover, we have $B_{li} A_{li}^* = 0$ from Lemma 2.2.5. Hence, we have

$$A_{li} A_{li}^* A_{li}^* = A_{li}^*. \quad (2.2.25)$$

From (2.2.24), we have

$$(A_{li}^* A_{li})(A_{li} A_{li}^*) = A_{li} A_{li}^*. \quad (2.2.26)$$

By taking $*$ on (2.2.25), we have

$$A_{li}A_{li}A_{li}^* = A_{li}. \quad (2.2.27)$$

Using this in (2.2.26), we have

$$A_{li}A_{li}^*A_{li} = A_{li}A_{li}^*, \quad (2.2.28)$$

and hence A_{li} is normal.

So, $A_{li} = A_{li}^*A_{li}A_{li}$ (from (2.2.24)) = $A_{li}A_{li}^*A_{li}$.

Therefore, A_{li} is a partial isometry which is normal and hence has same domain and range. \square

Lemma 2.2.9. :

B_{li} is a normal partial isometry and hence has same domain and range.

Proof : First we note that A_{ji} is a normal partial isometry and $A_{ji}B_{li} = 0$ for all $j \neq l$ (obtained from Lemma 2.2.6) implies that $\text{Ran}(A_{ji}^*) \subseteq \text{Ker}(B_{li}^*)$ and hence $\text{Ran}(A_{ji}) \subseteq \text{Ker}(B_{li}^*)$ which means $B_{li}^*A_{ji} = 0$ for all $j \neq l$.

To obtain $B_{li}^*B_{li}B_{li} = B_{li}$, we apply κ and multiply by B_{li} on the right of (2.2.16) and then use $A_{li}^*B_{li} = 0$ from Lemma 2.2.5, $A_{ji}B_{li} = 0$ for all $j \neq l$ (from Lemma 2.2.6 which implies $B_{li}^*A_{ji} = 0$ for all $j \neq l$ as above) and $B_{ji}B_{li} = 0$ for all $j \neq l$ from Lemma 2.2.6 .

Similarly, we have $B_{li}B_{li}^*B_{li} = B_{li}^*$ by applying κ and multiplying by B_{li}^* on the right of (2.2.12) obtained from Lemma 2.2.5 and then use $A_{li}B_{li}^* = 0$ (Lemma 2.2.5), $B_{li}A_{ji}^* = 0$ for all $j \neq l$ and $B_{li}B_{ji} = 0$ for all $j \neq l$ (Lemma 2.2.6).

Using $B_{li}^*B_{li}B_{li} = B_{li}$ and $B_{li}B_{li}^*B_{li} = B_{li}^*$ as in Lemma 2.2.8, we have B_{li} is a normal partial isometry. \square

Now, we use the condition $\alpha(U_i)\alpha(U_j) = \alpha(U_j)\alpha(U_i)$ for all i, j .

Lemma 2.2.10. :

For all $k \neq l$,

$$A_{ik}A_{jl} + A_{il}A_{jk} = A_{jl}A_{ik} + A_{jk}A_{il}, \quad (2.2.29)$$

$$A_{ik}B_{jl} + B_{il}A_{jk} = B_{jl}A_{ik} + A_{jk}B_{il}, \quad (2.2.30)$$

$$B_{ik}A_{jl} + A_{il}B_{jk} = A_{jl}B_{ik} + B_{jk}A_{il}, \quad (2.2.31)$$

$$B_{ik}B_{jl} + B_{il}B_{jk} = B_{jl}B_{ik} + B_{jk}B_{il}. \quad (2.2.32)$$

Proof : The result follows by equating the coefficients of $U_kU_l, U_kU_l^{-1}, U_k^{-1}U_l$ and $U_k^{-1}U_l^{-1}$ (where $k \neq l$) in $\alpha(U_i)\alpha(U_j) = \alpha(U_j)\alpha(U_i)$ for all i, j .

□

Lemma 2.2.11. :

$A_{ik}B_{jl} = B_{jl}A_{ik}$ for all $i \neq j, k \neq l$.

Proof : From Lemma 2.2.10, we have for all $k \neq l$, $A_{ik}B_{jl} - B_{jl}A_{ik} = A_{jk}B_{il} - B_{il}A_{jk}$. We consider the case where $i \neq j$.

We have, $\text{Ran}(A_{ik}B_{jl} - B_{jl}A_{ik}) \subseteq \text{Ran}(A_{ik}) + \text{Ran}(B_{jl}) \subseteq \text{Ran}(B_{jl}^*B_{jl} + A_{ik}^*A_{ik})$ (using the facts that A_{ik} and B_{jl} are normal partial isometries by Lemma 2.2.8 and 2.2.9 and also that $B_{jl}^*B_{jl}$ and $A_{ik}^*A_{ik}$ are projections).

Similarly, $\text{Ran}(A_{jk}B_{il} - B_{il}A_{jk}) \subseteq \text{Ran}(B_{il}^*B_{il} + A_{jk}^*A_{jk})$.

Let

$$T_1 = A_{ik}B_{jl} - B_{jl}A_{ik}, \quad (2.2.33)$$

$$T_2 = A_{jk}B_{il} - B_{il}A_{jk}, \quad (2.2.34)$$

$$T_3 = B_{jl}^*B_{jl} + A_{ik}^*A_{ik}, \quad (2.2.35)$$

$$T_4 = B_{il}^*B_{il} + A_{jk}^*A_{jk}. \quad (2.2.36)$$

Hence, $T_1 = T_2$, $\text{Ran}T_1 \subseteq \text{Ran}T_3$, $\text{Ran}T_2 \subseteq \text{Ran}T_4$.

We claim that $T_4T_3 = 0$.

Then $\text{Ran}(T_3) \subseteq \text{Ker}(T_4)$.

But $\text{Ran}T_1 \subseteq \text{Ran}T_3$ will imply that $\text{Ran}T_1 \subseteq \text{Ker}T_4$. Hence, $\text{Ran}(T_2) \subseteq \text{Ker}(T_4) = \overline{\text{Ran}(T_4^*)}^\perp = \overline{\text{Ran}(T_4)}^\perp$. But $\text{Ran}(T_2) \subseteq \text{Ran}(T_4)$ which implies that $\text{Ran}(T_2) = 0$ and hence both T_2 and T_1 are zero. Thus, the proof of the lemma will be complete if we can prove the claim.

$$T_4T_3 = (B_{il}^*B_{il} + A_{jk}^*A_{jk})(B_{jl}^*B_{jl} + A_{ik}^*A_{ik}) \quad (2.2.37)$$

$$= B_{il}^*B_{il}B_{jl}^*B_{jl} + B_{il}^*B_{il}A_{ik}^*A_{ik} + A_{jk}^*A_{jk}B_{jl}^*B_{jl} + A_{jk}^*A_{jk}A_{ik}^*A_{ik}. \quad (2.2.38)$$

From Lemma 2.2.6, we have for all $i \neq j$, $B_{il}B_{jl} = 0$ implying $B_{il}B_{jl}^* = 0$ as B_{jl} is a normal partial isometry.

Again, from Lemma 2.2.7 for all $k \neq l$, $B_{il}A_{ik} = 0$. Then A_{ik} is a normal partial isometry implies that $B_{il}A_{ik}^* = 0$ for all $k \neq l$.

Similarly, by taking adjoint of the relation $B_{jl}A_{jk}^* = 0$ for all $k \neq l$ obtained from Lemma 2.2.7, we have $A_{jk}B_{jl}^* = 0$.

From Lemma 2.2.6, we have $A_{jk}A_{ik} = 0$ for all $i \neq j$. A_{ik} is a normal partial isometry implies that $A_{jk}A_{ik}^* = 0$ for all $i \neq j$.

Using these, we note that $T_4T_3 = 0$ which proves the claim and hence the lemma. □

Lemma 2.2.12. :

$$A_{ik}B_{jk} = 0 = B_{jk}A_{ik}, \quad (2.2.39)$$

$$A_{ki}B_{kj} = 0 = B_{kj}A_{ki} \quad (2.2.40)$$

for all $i \neq j$ and for all k .

Proof : By Lemma 2.2.6, we have $A_{ik}B_{jk} = 0$ and $B_{jk}A_{ik}^* = 0$ for all $i \neq j$. The second relation along with the fact that A_{ik} is a normal partial isometry implies that $B_{jk}A_{ik} = 0$ for all $i \neq j$.

Thus, $A_{ik}B_{jk} = 0 = B_{jk}A_{ik}$ for all $i \neq j$.

Applying κ on the above equation and using B_{kj} and A_{ki} are normal partial isometries, we have $A_{ki}B_{kj} = 0 = B_{kj}A_{ki}$. □

Lemma 2.2.13. : $A_{ik}B_{ik} = B_{ik}A_{ik}$ for all i, k .

Proof : We have $A_{ij}^*B_{ij} = 0 = B_{ij}^*A_{ij}$ from Lemma 2.2.5. Using the fact that B_{ij} and A_{ij} are normal partial isometry we have $A_{ij}^*B_{ij}^* = 0 = B_{ij}^*A_{ij}^*$ and hence $A_{ij}B_{ij} = B_{ij}A_{ij}$. □

Lemma 2.2.14. :

$A_{ik}A_{jl} = A_{jl}A_{ik}$ for all $i \neq j, k \neq l$.

Proof : Using (2.2.29) in Lemma 2.2.10, we proceed as in Lemma 2.2.11 to get $\text{Ran}(A_{ik}A_{jl} - A_{jl}A_{ik}) \subseteq \text{Ran}(A_{jl}A_{jl}^* + A_{ik}A_{ik}^*)$ and $\text{Ran}(A_{jk}A_{il} - A_{il}A_{jk}) \subseteq \text{Ran}(A_{il}A_{il}^* + A_{jk}A_{jk}^*)$.

We claim that $(A_{ik}A_{ik}^* + A_{jl}A_{jl}^*)(A_{jk}A_{jk}^* + A_{il}A_{il}^*) = 0$.

Then by the same reasonings as given in Lemma 2.2.11 we will have : $A_{jk}A_{il} = A_{il}A_{jk}$.

To prove the claim, we use $A_{ik}A_{jk} = 0$ for all $i \neq j$ from Lemma 2.2.6 (which implies $A_{jk}^*A_{ik} = 0$ for all $i \neq j$ as A_{ik} is a normal partial isometry), $A_{il}^*A_{ik} = 0$ for all $k \neq l$ from Lemma 2.2.7 (which implies $A_{il}^*A_{ik} = 0$ for all $k \neq l$ as A_{ik} is a normal partial isometry) and $A_{il}A_{jl} = 0$ for all $i \neq j$ from Lemma 2.2.6 (which implies $A_{jl}^*A_{il} = 0$ for all $i \neq j$ as A_{il}^* is a normal partial isometry). □

Lemma 2.2.15. :

$$A_{ik}A_{il} = A_{il}A_{ik} \text{ for all } k \neq l, \quad (2.2.41)$$

$$A_{ik}A_{jk} = A_{jk}A_{ik} \text{ for all } i \neq j. \quad (2.2.42)$$

Proof : From Lemma 2.2.6, we have $A_{ki}A_{li} = 0$ for all $k \neq l$.

Applying κ and taking adjoint, we have $A_{ik}A_{il} = 0$ for all $k \neq l$. Interchanging k and l , we get $A_{il}A_{ik} = 0$ for all $k \neq l$. Hence, $A_{ik}A_{il} = A_{il}A_{ik}$ for all $k \neq l$.

From Lemma 2.2.6, we have $A_{ik}A_{jk} = 0$ for all $i \neq j$. Interchanging i and j , we have $A_{jk}A_{ik} = 0$ for all $i \neq j$. \square

Remark 2.2.16. *Proceeding in an exactly similar way, we have that B_{ij} 's commute among themselves.*

Theorem 2.2.17. *The Quantum isometry group of $C(\mathbb{T}^n)$ is commutative as a C^* algebra and hence coincides with the classical isometry group $C(\mathbb{T}^n \rtimes (\mathbf{Z}_2^n \rtimes S_n))$.*

Proof : Follows from the results in lemma 2.2.11 - 2.2.15 and the remark following them. \square

Chapter 3

Quantum group of orientation preserving Riemannian isometries

3.1 Introduction

The formulation of quantum isometry groups in [30] had a major drawback from the viewpoint of noncommutative geometry since it needed a ‘good’ Laplacian to exist. In noncommutative geometry it is not always easy to verify such an assumption about the Laplacian, and thus it would be more appropriate to have a formulation in terms of the Dirac operator directly. This is what we aim to achieve in the present chapter.

The group of Riemannian isometries of a compact Riemannian manifold M can be viewed as the universal object in the category of all compact metrizable groups acting on M , with smooth and isometric action. Moreover, let us assume that the manifold has a spin structure (hence in particular orientable, so we can fix a choice of orientation) and D denotes the conventional Dirac operator acting as an unbounded self-adjoint operator on the Hilbert space \mathcal{H} of square integrable spinors. Then, it can be proved that a group action on the manifold lifts as a unitary representation on the Hilbert space \mathcal{H} which commutes with D if and only if the action on the manifold is an orientation preserving isometric action. Therefore, to define the quantum analogue of the group of orientation-preserving Riemannian isometries of a possibly noncommutative manifold given by a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, it is reasonable to consider a category \mathbf{Q}' of compact quantum groups having unitary (co-) representation, say U , on \mathcal{H} , which commutes with D , and the action on $\mathcal{B}(\mathcal{H})$ obtained by conjugation maps \mathcal{A}^∞ into its weak closure. A universal object in this category, if it exists, should define the ‘quantum group of orientation preserving Riemannian isometries’ of the underlying spectral triple. It is easy to see that any object (\mathcal{S}, U) of the category \mathbf{Q}' gives an equivariant spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ with respect to the action of \mathcal{S} implemented by U . It may be noted

that recently there has been a lot of interest and work (see, for example, [13], [18], [23]) towards construction of quantum group equivariant spectral triples. In all these works, given a C^* -subalgebra \mathcal{A} of $\mathcal{B}(\mathcal{H})$ and a CQG \mathcal{Q} having a unitary representation U on \mathcal{H} such that $\alpha_U(\equiv \text{ad}_U)$ gives an action of \mathcal{Q} on \mathcal{A} , the authors investigate the possibility of constructing a (nontrivial) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ on a suitable dense subalgebra \mathcal{A}^∞ of \mathcal{A} such that U commutes with $D \otimes I$, that is, D is equivariant. Our interest here is in the (sort of) converse direction: given a spectral triple, we want to consider all possible CQG representations with respect to which the spectral triple is equivariant; and if there exists a universal object in the corresponding category, that is, \mathbf{Q}' , we should call it the quantum group of orientation preserving isometries.

Unfortunately, even in the finite-dimensional (but with noncommutative \mathcal{A}) situation this category may often fail to have a universal object, as will be discussed later. It turns out, however, that if we fix a suitable densely defined (in the WOT) functional on $\mathcal{B}(\mathcal{H})$ (to be interpreted as the choice of a ‘volume form’) then there exists a universal object in the subcategory of \mathbf{Q}' obtained by restricting the object-class to the quantum group actions which also preserve the given functional. The subtle point to note here is that unlike the classical group actions on $\mathcal{B}(\mathcal{H})$ which always preserve the usual trace, a quantum group action may not do so. In fact, it was proved by Goswami in [31] that given an object (\mathcal{Q}, U) of \mathbf{Q}' (where \mathcal{Q} is a compact quantum group and U denotes its unitary co-representation on \mathcal{H}), we can find a positive invertible operator R in \mathcal{H} so that the given spectral triple is R -twisted in the sense of [31] and the corresponding functional τ_R (which typically differs from the usual trace of $\mathcal{B}(\mathcal{H})$ and can have a nontrivial modularity) is preserved by the action of \mathcal{Q} . This makes it quite natural to work in the setting of twisted spectral data (as defined in [31]).

Motivated by the ideas of Woronowicz and Soltan ([68], [53]), we actually consider a bigger category called the category of ‘quantum families of smooth orientation preserving Riemannian isometries’. The underlying C^* -algebra of the quantum group of orientation preserving isometries (whenever exists) has been identified with the universal object in this bigger category and moreover, it is shown to be equipped with a canonical coproduct making it into a compact quantum group.

In this chapter, we discuss a number of examples, covering the classical spectral triple on $C^\infty(\mathbb{T}^2)$ as well as the equivariant spectral triples constructed recently on $SU_\mu(2)$. It may be relevant to point out here that it was not clear whether one could accommodate the spectral triples on $SU_\mu(2)$ and the Podles’ spheres $S_{\mu,c}^2$ in the framework of [30], since it is very difficult to give a nice description of the space of ‘noncommutative’ forms and the Laplacian for these examples. However, the present formulation in terms of the Dirac operator makes it easy to accommodate them, and we have been able to identify $U_\mu(2)$ and $SO_\mu(3)$ as the universal quantum group of orientation preserving isometries

for the spectral triples on $SU_\mu(2)$ and $S^2_{\mu,c}$ respectively (the computations for $S^2_{\mu,c}$ have been presented in Chapter 5).

We conclude this section with an important remark about the use of the phrase ‘orientation -preserving’ in our terminology. We recall from Remark 1.5.5 that by a ‘classical spectral triple’ we always mean the spectral triple obtained by the Dirac operator on the spinors. This is absolutely crucial in view of the fact that the Hodge Dirac operator $d + d^*$ on the L^2 -space of differential forms also gives a spectral triple of compact type on any compact Riemannian (not necessarily with a spin structure) manifold M , but the action of the full isometry group $ISO(M)$ (and not just the subgroup of orientation preserving isometries $ISO^+(M)$, even when M is orientable) lifts to a canonical unitary representation on this space commuting with $d + d^*$. In fact, the category of groups acting on M such that the action comes from a unitary representation commuting with $d + d^*$, has $ISO(M)$, and not $ISO^+(M)$, as its universal object. So, one must stick to the Dirac operator on spinors to obtain the group of orientation preserving isometries in the usual geometric sense. This also has a natural quantum generalization, as we shall see in section 3.3.

3.2 Definition and existence of the quantum group of orientation-preserving isometries

3.2.1 The classical case

We first discuss the classical situation clearly, which will serve as a motivation for our quantum formulation.

We begin with a few basic facts about topologizing the space $C^\infty(M, N)$ where M, N are smooth manifolds. Let Ω be an open set of \mathbb{R}^n . We endow $C^\infty(\Omega)$ with the usual Frechet topology coming from uniform convergence (over compact subsets) of partial derivatives of all orders. The space $C^\infty(\Omega)$ is complete with respect to this topology, so is a Polish space in particular. Moreover, by the Sobolev imbedding theorem (Corollary 1.21, [48]), $\cap_{k \geq 0} H_k(\Omega) = C^\infty(\Omega)$ as a set, where $H_k(\Omega)$ denotes the k -th Sobolev space. Thus, $C^\infty(\Omega)$ has also the Hilbertian seminorms coming from the Sobolev spaces, hence the corresponding Frechet topology. We claim that these two topologies on $C^\infty(\Omega)$ coincide. Indeed, the inclusion map from $C^\infty(\Omega)$ into $\cap_k H_k(\Omega)$, is continuous and surjective, so by the open mapping theorem for Frechet space, the inverse is also continuous, proving our claim.

Given two second countable smooth manifolds M, N , we shall equip $C^\infty(M, N)$ with the weakest locally convex topology making $C^\infty(M, N) \ni \phi \mapsto f \circ \phi \in C^\infty(M)$ Frechet continuous for every f in $C^\infty(N)$.

For topological or smooth fibre or principal bundles E, F over a second countable smooth manifold M , we shall denote by $\text{Hom}(E, F)$ the set of bundle morphisms from E to F . We remark that the total space of a locally trivial topological bundle such that the base and the fibre spaces are locally compact Hausdorff second countable must itself be so, hence in particular Polish (that is, a complete separable metric space).

In particular, if E, F are locally trivial principal G -bundles over a common base, such that the (common) base as well as structure group G are locally compact Hausdorff and second countable, then $\text{Hom}(E, F)$ is a Polish space.

We need a standard fact, stated below as Lemma 3.2.2, about the measurable lift of Polish space valued functions.

Before that, we introduce some notions.

A multifunction $G : X \rightarrow Y$ is a map with domain X and whose values are nonempty subsets of Y . For $A \subseteq Y$, we put $G^{-1}(A) = \{x \in X : G(x) \cap A \neq \emptyset\}$.

A selection of a multifunction $G : X \rightarrow Y$ is a point map $s : X \rightarrow Y$ such that $s(x)$ belongs to $G(x)$ for all x in X . Now let Y be a Polish space and σ_X a σ -algebra on X . A multifunction $G : X \rightarrow Y$ is called σ_X measurable if $G^{-1}(U)$ belongs to σ_X for every open set U in Y .

The following well known selection theorem is Theorem 5.2.1 of [55] and was proved by Kuratowski and Ryll-Nardzewski.

Proposition 3.2.1. *Let σ_X be a σ algebra on X and Y a Polish space. Then, every σ_X measurable, closed valued multifunction $F : X \rightarrow Y$ admits a σ_X measurable selection.*

A trivial consequence of this result is the following:

Lemma 3.2.2. *Let M be a compact metrizable space, B, \tilde{B} Polish spaces such that there is an n -covering map $\Lambda : \tilde{B} \rightarrow B$. Then any continuous map $\xi : M \rightarrow B$ admits a lifting $\tilde{\xi} : M \rightarrow \tilde{B}$ which is Borel measurable and $\Lambda \circ \tilde{\xi} = \xi$. In particular, if \tilde{B} and B are topological bundles over M , with Λ being a bundle map, any continuous section of B admits a lifting which is a measurable section of \tilde{B} .*

We shall now give an operator-theoretic characterization of the classical group of orientation-preserving Riemannian isometries, which will be the motivation of our definition of its quantum counterpart. Let M be a compact Riemannian n dimensional spin manifold, with a fixed choice of orientation. We recall the notations as in subsection 1.4.3. In particular, the spinor bundle S is the associated bundle of a principal $\text{Spin}(n)$ -bundle, say P , on M which has a canonical 2-covering bundle-map Λ from P to the frame-bundle F (which is an $SO(n)$ -principal bundle), such that locally Λ is of the form $(\text{id}_M \otimes \lambda)$ where λ is the two covering map from $\text{Spin}(n)$ to $SO(n)$. Moreover, the spinor space will be denoted by Δ_n . Let f be a smooth orientation preserving Riemannian isometry of M , and consider the bundles $E = \text{Hom}(F, f^*(F))$ and

$\tilde{E} = \text{Hom}(P, f^*(P))$ (where Hom denotes the set of bundle maps). We view df as a section of the bundle E in the natural way. By the Lemma 3.2.2 we obtain a measurable lift $\tilde{d}f : M \rightarrow \tilde{E}$, which is a measurable section of \tilde{E} . Using this, we define a map on the space of measurable section of $S = P \times_{\text{Spin}(n)} \Delta_n$ as follows: given a (measurable) section ξ of S , say of the form $\xi(m) = [p(m), v]$, with $p(m)$ in P_m, v in Δ_n , we define $U\xi$ by $(U\xi)(m) = [\tilde{d}f(f^{-1}(m))(p(f^{-1}(m))), v]$. Note that sections of the above form constitute a total subset in $L^2(S)$, and the map $\xi \mapsto U\xi$ is clearly a densely defined linear map on $L^2(S)$, whose fibre-wise action is unitary since the $\text{Spin}(n)$ action is so on Δ_n . Thus it extends to a unitary U on $\mathcal{H} = L^2(S)$. Any such U , induced by the map f , will be denoted by U_f (it is not unique since the choice of the lifting used in its construction is not unique).

Theorem 3.2.3. *Let M be a compact Riemannian spin manifold (hence orientable , and fix a choice of orientation) with the usual Dirac operator D acting as an unbounded self-adjoint operator on the Hilbert space \mathcal{H} of the square integrable spinors, and let S denote the spinor bundle, with $\Gamma(S)$ being the $C^\infty(M)$ module of smooth sections of S . Let $f : M \rightarrow M$ be a smooth one-to-one map which is a Riemannian orientation preserving isometry. Then the unitary U_f on \mathcal{H} commutes with D and $U_f M_\phi U_f^* = M_{\phi \circ f}$, for any ϕ in $C(M)$, where M_ϕ denotes the operator of multiplication by ϕ on $L^2(S)$. Moreover, when the dimension of M is even, U_f commutes with the canonical grading γ on $L^2(S)$.*

Conversely, suppose that U is a unitary on \mathcal{H} such that $UD = DU$ and the map $\alpha_U(X) = UXU^{-1}$ for X in $\mathcal{B}(\mathcal{H})$ maps $\mathcal{A} = C(M)$ into $L^\infty(M) = \mathcal{A}''$, then there is a smooth one-to-one orientation-preserving Riemannian isometry f on M such that $U = U_f$. We have the same result in the even case, if we assume furthermore that $U\gamma = \gamma U$.

Proof: From the construction of U_f , it is clear that $U_f M_\phi U_f^{-1} = M_{\phi \circ f}$. Moreover, since the Dirac operator D commutes with the $\text{Spin}(n)$ -action on S , we have $U_f D = D U_f$ on each fibre, hence on $L^2(S)$. In the even dimensional case, it is easy to see that the $\text{Spin}(n)$ action commutes with γ (the grading operator), hence U_f does so.

For the converse, first note that α_U is a unital $*$ -homomorphism on $L^\infty(M, d\text{vol})$ and thus must be of the form $\psi \mapsto \psi \circ f$ for some measurable f . We claim that f must be smooth. Fix any smooth g on M and consider $\phi = g \circ f$. We have to argue that ϕ is smooth. Let δ_D denote the generator of the strongly continuous one-parameter group of automorphism $\beta_t(X) = e^{itD} X e^{-itD}$ on $\mathcal{B}(\mathcal{H})$ (with respect to the weak operator topology, say). From the assumption that D and U commute it is clear that α_U maps $\mathcal{D} := \bigcap_{n \geq 1} \text{Dom}(\delta_D^n)$ into itself and since $C^\infty(M) \subset \mathcal{D}$, we conclude that $\alpha_U(M_\phi) = M_{\phi \circ g}$ belongs to \mathcal{D} . We claim that this implies the smoothness of ϕ . Let m belongs to M and choose a local chart (V, ψ) at m , with the coordinates (x_1, \dots, x_n) , such

that $\Omega = \psi(V) \subseteq \mathbb{R}^n$ has compact closure, $S|_V$ is trivial and D has the local expression $D = i \sum_{j=1}^n \mu(e_j) \nabla_j$, where $\nabla_j = \nabla_{\frac{\partial}{\partial x_j}}$ denotes the covariant derivative (with respect to the canonical Levi Civita connection) operator along the vector field $\frac{\partial}{\partial x_j}$ on $L^2(\Omega)$ and $\mu(v)$ denotes the Clifford multiplication by a vector v . Now, $\phi \circ \psi^{-1} \in L^\infty(\Omega) \subseteq L^2(\Omega)$ and it is easy to observe from the above local structure of D that $[D, M_\phi]$ has the local expression $\sum_j i M_{\frac{\partial}{\partial x_j} \phi} \otimes \mu(e_j)$. Thus, the fact $M_\phi \in \bigcap_{n \geq 1} \text{Dom}(\delta_D^n)$ implies $\phi \circ \psi^{-1}$ is in $\text{Dom}(d_{j_1} \dots d_{j_k})$ for every integer tuples (j_1, \dots, j_k) , $j_i \in \{1, \dots, n\}$, where $d_j := \frac{\partial}{\partial x_j}$. In other words, $\phi \circ \psi^{-1}$ is in $H^k(\Omega)$ for all $k \geq 1$, where $H^k(\Omega)$ denotes the k -th Sobolev space on Ω (see [48]). By Sobolev's Theorem (see, for example. [48], Corollary 1.21, page 24) it follows that $\phi \circ \psi^{-1}$ is in $C^\infty(\Omega)$.

We note that f is one-to-one as $\phi \rightarrow \phi \circ f$ is an automorphism of L^∞ . Now, we shall show that f is an isometry of the metric space (M, d) , where d is the metric coming from the Riemannian structure, and we have the explicit formula (1.4.1)

$$d(p, q) = \sup_{\phi \in C^\infty(M), \|[D, M_\phi]\| \leq 1} |\phi(p) - \phi(q)|.$$

Since U commutes with D , we have $\|[D, M_{\phi \circ f}]\| = \|[D, U M_\phi U^*]\| = \|U[D, M_\phi]U^*\| = \|[D, M_\phi]\|$ for every ϕ , from which it follows that $d(f(p), f(q)) = d(p, q)$. Finally, f is orientation preserving if and only if the volume form (say ω), which defines the choice of orientation, is preserved by the natural action of df , that is, $(df \wedge \dots \wedge df)(\omega) = \omega$. This will follow from the explicit description of ω in terms of D , given by (see [65] Page 26, also see [20])

$$\omega(\phi_0 d\phi_1 \dots d\phi_n) = \tau(\epsilon M_{\phi_0} [D, M_{\phi_1}] \dots [D, M_{\phi_n}]),$$

where ϕ_0, \dots, ϕ_n belong to $C^\infty(M)$, $\epsilon = 1$ in the odd case and $\epsilon = \gamma$ (the grading operator) in the even case and τ denotes the volume integral. In fact, $\tau(X) = \lim_{t \rightarrow 0^+} \frac{\text{Tr}(e^{-tD^2} X)}{\text{Tr}(e^{-tD^2})}$ (where Lim is as in subsection 1.5.2), which implies $\tau(UXU^*) = \tau(X)$ for all X in $\mathcal{B}(\mathcal{H})$ (using the fact that D and U commute). Thus,

$$\begin{aligned} & \omega(\phi_0 \circ f \, d(\phi_1 \circ f) \dots d(\phi_n \circ f)) \\ &= \tau(\epsilon U M_{\phi_0} U^* U [D, M_{\phi_1}] U^* \dots U [D, M_{\phi_n}] U^*) \\ &= \tau(U \epsilon M_{\phi_0} [D, M_{\phi_1}] \dots [D, M_{\phi_n}] U^*) \\ &= \tau(\epsilon M_{\phi_0} [D, M_{\phi_1}] \dots [D, M_{\phi_n}]) \\ &= \omega(\phi_0 d\phi_1 \dots d\phi_n). \end{aligned}$$

□

Now we turn to the case of a family of maps. We are ready to state and prove the operator-theoretic characterization of ‘set of orientation-preserving isometries’.

Theorem 3.2.4. *Let X be a compact metrizable space and $\psi : X \times M \rightarrow M$ is a map such that ψ_x defined by $\psi_x(m) = \psi(x, m)$ is a smooth orientation preserving Riemannian isometry and $x \mapsto \psi_x \in C^\infty(M, M)$ is continuous with respect to the locally convex topology of $C^\infty(M, M)$ mentioned before.*

Then there exists a ($C(X)$ -linear) unitary U_ψ on the Hilbert $C(X)$ -module $\mathcal{H} \otimes C(X)$ (where $\mathcal{H} = L^2(S)$ as in Theorem 3.2.3) such that for all x belonging to X , $U_x := (\text{id} \otimes \text{ev}_x)U_\psi$ is a unitary of the form U_{ψ_x} on the Hilbert space \mathcal{H} commuting with D and $U_x M_\phi U_x^{-1} = M_{\phi \circ \psi_x^{-1}}$. If in addition, the manifold is even dimensional, then U_{ψ_x} commutes with the grading operator γ .

Conversely, if there exists a $C(X)$ -linear unitary U on $\mathcal{H} \otimes C(X)$ such that $U_x := (\text{id} \otimes \text{ev}_x)(U)$ is a unitary commuting with D for all x , (and U_x commutes with the grading operator γ if the manifold is even dimensional) and $(\text{id} \otimes \text{ev}_x)\alpha_U(L^\infty(M)) \subseteq L^\infty(M)$ for all x in X , then there exists a map $\psi : X \times M \rightarrow M$ satisfying the conditions mentioned above such that $U = U_\psi$.

Proof: Consider the bundles $\hat{F} = X \times F$ and $\hat{P} = X \times P$ over $X \times M$, with fibres at (x, m) isomorphic with F_m and P_m respectively, and where F and P are respectively the bundles of orthonormal frames and the $\text{Spin}(n)$ bundle discussed before. Moreover, denote by Ψ the map from $X \times M$ to itself given by $(x, m) \mapsto (x, \psi(x, m))$. Let $\pi_X : \text{Hom}(\hat{F}, \Psi^*(\hat{F})) \rightarrow X$ be the obvious map obtained by composing the projection map of the $X \times M$ bundle with the projection from $X \times M$ to X , and let us denote by B the closed subset of the Polish space $C(X, \text{Hom}(\hat{F}, \Psi^*(\hat{F})))$ consisting of those f such that for all x , $\pi_X(f(x)) = x$. Define \tilde{B} in a similar way replacing \hat{F} by \hat{P} . The covering map from P to F induces a covering map from \tilde{B} to B as well. Let $d'_\psi : M \rightarrow B$ be the map given by $d'_\psi(m)(x) \equiv d'_\psi(x, m) = d\psi_x|_m$. Then by Lemma 3.2.2 there exists a measurable lift of d'_ψ , say $\widetilde{d'_\psi}$ from M into \tilde{B} . Since $d'_\psi(x, m) \in \text{Hom}(F_m, F_{\psi(x, m)})$, it is clear that the lift $\widetilde{d'_\psi}(x, m)$ will be an element of $\text{Hom}(P_m, P_{\psi(x, m)})$.

We can identify $\mathcal{H} \otimes C(X)$ with $C(X \rightarrow \mathcal{H})$, and since \mathcal{H} has a total set \mathcal{F} (say) consisting of sections of the form $[p(\cdot), v]$, where $p : M \rightarrow P$ is a measurable section of P and v belongs to Δ_n , we have a total set $\tilde{\mathcal{F}}$ of $\mathcal{H} \otimes C(X)$ consisting of \mathcal{F} valued continuous functions from X . Any such function can be written as $[\Xi, v]$ with $\Xi : X \times M \rightarrow P$, $v \in \Delta_n$, and $\Xi(x, m) \in P_m$, and we define U on $\tilde{\mathcal{F}}$ by $U[\Xi, v] = [\Theta, v]$, where

$$\Theta(x, m) = \widetilde{d'_\psi}(x, \psi_x^{-1}(m))(\Xi(x, \psi_x^{-1}(m))).$$

It is clear from the construction of the lift that U is indeed a $C(X)$ -linear isometry which maps the total set $\tilde{\mathcal{F}}$ onto itself, so extends to a unitary on the whole of $\mathcal{H} \otimes C(X)$

with the desired properties.

Conversely, given U as in the statement of the converse part of the theorem, we observe that for each x in X , by Theorem 3.2.3, $(\text{id} \otimes \text{ev}_x)U = U_{\psi_x}$ for some ψ_x such that ψ_x is a smooth orientation preserving Riemannian isometry. This defines the map ψ by setting $\psi(x, m) = \psi_x(m)$. The proof will be complete if we can show that $x \mapsto \psi_x \in C^\infty(M, M)$ is continuous, which is equivalent to showing that whenever $x_n \rightarrow x$ in the topology of X , we must have $\phi \circ \psi_{x_n} \rightarrow \phi \circ \psi_x$ in the Frechet topology of $C^\infty(M)$, for any $\phi \in C^\infty(M)$. However, by Lemma 1.1.8, we have $(\text{id} \otimes \text{ev}_{x_n})\alpha_U([D, M_\phi]) \rightarrow (\text{id} \otimes \text{ev}_x)\alpha_U([D, M_\phi])$ in the strong operator topology where $\alpha_U(X) = UXU^{-1}$. Since U commutes with D , this implies

$$(\text{id} \otimes \text{ev}_{x_n})[D \otimes \text{id}, \alpha_U(M_\phi)] \rightarrow (\text{id} \otimes \text{ev}_x)[D \otimes \text{id}, \alpha_U(M_\phi)],$$

that is, for all ξ in $L^2(S)$,

$$[D, M_{\phi \circ \psi_{x_n}}]\xi \xrightarrow{L^2} [D, M_{\phi \circ \psi_x}]\xi.$$

By choosing ϕ with support in a local trivializing coordinate neighborhood for S , and then using the local expression of D used in the proof of Theorem 3.2.3, we conclude that $d_k(\phi \circ \psi_{x_n}) \xrightarrow{L^2} d_k(\phi \circ \psi_x)$ (where d_k is as in the proof of Theorem 3.2.3). Similarly, by taking repeated commutators with D , we can show the L^2 convergence with d_k replaced by $d_{k_1} \dots d_{k_m}$ for any finite tuple (k_1, \dots, k_m) . In other words, $\phi \circ \psi_{x_n} \rightarrow \phi \circ \psi_x$ in the topology of $C^\infty(M)$ described before. \square

3.2.2 Quantum group of orientation-preserving isometries of an R -twisted spectral triple

In view of the characterization of orientation-preserving isometric action on a classical manifold (Theorem 3.2.4), we give the following definitions.

Definition 3.2.5. A quantum family of orientation preserving isometries for the (odd, compact type) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ is given by a pair (\mathcal{S}, U) where \mathcal{S} is a separable unital C^* -algebra and U is a linear map from \mathcal{H} to $\mathcal{H} \otimes \mathcal{S}$ such that \tilde{U} given by $\tilde{U}(\xi \otimes b) = U(\xi)(1 \otimes b)$ (ξ in \mathcal{H} , b in \mathcal{S}) extends to a unitary element of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{S})$ satisfying the following:

- (i) for every state ϕ on \mathcal{S} we have $U_\phi D = D U_\phi$, where $U_\phi := (\text{id} \otimes \phi)(\tilde{U})$;
- (ii) $(\text{id} \otimes \phi) \circ \alpha_U(a) \in (\mathcal{A}^\infty)''$ for all a in \mathcal{A}^∞ and state ϕ on \mathcal{S} , where $\alpha_U(x) := \tilde{U}(x \otimes 1)\tilde{U}^*$ for x belonging to $\mathcal{B}(\mathcal{H})$.

In case the C^* -algebra \mathcal{S} has a coproduct Δ such that (\mathcal{S}, Δ) is a compact quantum group and U is a unitary representation of (\mathcal{S}, Δ) on \mathcal{H} , we say that (\mathcal{S}, Δ) acts by

orientation-preserving isometries on the spectral triple.

In case the spectral triple is even with the grading operator γ , a quantum family of orientation preserving isometries $(\mathcal{A}^\infty, \mathcal{H}, D, \gamma)$ will be defined exactly as above, with the only extra condition being that U commutes with γ .

From now on, we will mostly consider odd spectral triples. However let us remark that in the even case, all the definitions and results obtained by us will go through with some obvious modifications. We also remark that all our spectral triples are of compact type.

Consider the category $\mathbf{Q} \equiv \mathbf{Q}(\mathcal{A}^\infty, \mathcal{H}, D) \equiv \mathbf{Q}(D)$ with the object-class consisting of all quantum families of orientation preserving isometries (\mathcal{S}, U) of the given spectral triple, and the set of morphisms $\text{Mor}((\mathcal{S}, U), (\mathcal{S}', U'))$ being the set of unital C^* -homomorphisms $\Phi : \mathcal{S} \rightarrow \mathcal{S}'$ satisfying $(\text{id} \otimes \Phi)(U) = U'$. We also consider another category $\mathbf{Q}' \equiv \mathbf{Q}'(\mathcal{A}^\infty, \mathcal{H}, D) \equiv \mathbf{Q}'(D)$ whose objects are triplets (\mathcal{S}, Δ, U) , where (\mathcal{S}, Δ) is a compact quantum group acting by orientation preserving isometries on the given spectral triple, with U being the corresponding unitary representation. The morphisms are the homomorphisms of compact quantum groups which are also morphisms of the underlying quantum families of orientation preserving isometries. The forgetful functor $F : \mathbf{Q}' \rightarrow \mathbf{Q}$ is clearly faithful, and we can view $F(\mathbf{Q}')$ as a subcategory of \mathbf{Q} .

Unfortunately, in general \mathbf{Q}' or \mathbf{Q} will not have a universal object. It is easily seen by taking the standard example $\mathcal{A}^\infty = M_n(\mathbb{C})$, $\mathcal{H} = \mathbb{C}^n$, $D = I$. Any CQG having a unitary representation on \mathbb{C}^n is an object of $\mathbf{Q}'(M_n(\mathbb{C}), \mathbb{C}^n, I)$. But by Proposition 1.2.34, there is no universal object in this category. However, the fact that comes to our rescue is that a universal object exists in each of the subcategories which correspond to the CQG actions preserving a given faithful functional on M_n .

On the other hand, given any equivariant spectral triple, it has been shown in [31] that there is a (not necessarily unique) canonical faithful functional which is preserved by the CQG action. For readers' convenience, we state this result (in a form suitable to us) briefly here. Before that, let us recall the definition of an R -twisted spectral data from [31].

Definition 3.2.6. *An R -twisted spectral data (of compact type) is given by a quadruplet $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ where*

1. *($\mathcal{A}^\infty, \mathcal{H}, D$) is a spectral triple of compact type.*
2. *R a positive (possibly unbounded) invertible operator such that R commutes with D .*
3. *For all $s \in \mathbb{R}$, the map $a \mapsto \sigma_s(a) := R^{-s}aR^s$ gives an automorphism of \mathcal{A}^∞ (not necessarily $*$ -preserving) satisfying $\sup_{s \in [-n, n]} \|\sigma_s(a)\| < \infty$ for all positive integer n .*

We shall also sometimes refer to $(\mathcal{A}^\infty, \mathcal{H}, D)$ as an R -twisted spectral triple.

Proposition 3.2.7. *Given a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ (of compact type) which is \mathcal{Q} -equivariant with respect to a representation of a CQG \mathcal{Q} on \mathcal{H} , we can construct a positive (possibly unbounded) invertible operator R on \mathcal{H} such that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is a twisted spectral data, and moreover, we have*

α_U preserves the functional τ_R defined at least on a weakly dense $$ -subalgebra \mathcal{E}_D of $\mathcal{B}(\mathcal{H})$ generated by the rank-one operators of the form $|\xi\rangle\langle\eta|$ where ξ, η are eigenvectors of D , given by*

$$\tau_R(x) = \text{Tr}(Rx), \quad x \in \mathcal{E}_D.$$

Remark 3.2.8. *When the Haar state of \mathcal{Q} is tracial, then it follows from the definition of R in Lemma 3.1 of [31] and Theorem 1.5 part 1. of [67] that R can be chosen to be I .*

Remark 3.2.9. *If V_λ denotes the eigenspace of D corresponding to the eigenvalue, say λ , it is clear that $\tau_R(X) = e^{t\lambda^2} \text{Tr}(Re^{-tD^2}X)$ for all $X = |\xi\rangle\langle\eta|$ with ξ, η belonging to V_λ and for any $t > 0$. Thus, the α_U -invariance of the functional τ_R on \mathcal{E}_D is equivalent to the α_U -invariance of the functional $X \mapsto \text{Tr}(XRe^{-tD^2})$ on \mathcal{E}_D for each $t > 0$. If, furthermore, the R -twisted spectral triple is Θ -summable in the sense that Re^{-tD^2} is trace class for every $t > 0$, the above is also equivalent to the α_U -invariance of the bounded normal functional $X \mapsto \text{Tr}(XRe^{-tD^2})$ on the whole of $\mathcal{B}(\mathcal{H})$. In particular, this implies that α_U preserves the functional $\mathcal{B}(\mathcal{H}) \ni x \mapsto \lim_{t \rightarrow 0+} \frac{\text{Tr}(xRe^{-tD^2})}{\text{Tr}(Re^{-tD^2})}$, where \lim is as defined in subsection 1.5.2.*

This motivates the following definition:

Definition 3.2.10. *Given an R -twisted spectral data $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ of compact type, a quantum family of orientation preserving isometries (\mathcal{S}, U) of $(\mathcal{A}^\infty, \mathcal{H}, D)$ is said to preserve the R -twisted volume, (simply said to be volume-preserving if R is understood) if one has $(\tau_R \otimes \text{id})(\alpha_U(x)) = \tau_R(x) \cdot 1_{\mathcal{S}}$ for all x in \mathcal{E}_D , where \mathcal{E}_D and τ_R are as in Proposition 3.2.7. We shall also call (\mathcal{S}, U) a quantum family of orientation-preserving isometries of the R -twisted spectral triple.*

If, furthermore, the C^ -algebra \mathcal{S} has a coproduct Δ such that (\mathcal{S}, Δ) is a CQG and U is a unitary representation of (\mathcal{S}, Δ) on \mathcal{H} , we say that (\mathcal{S}, Δ) acts by volume and orientation-preserving isometries on the R -twisted spectral triple.*

We shall consider the categories $\mathbf{Q}_R \equiv \mathbf{Q}_R(D)$ and $\mathbf{Q}'_R \equiv \mathbf{Q}'_R(D)$ which are the full subcategories of \mathbf{Q} and \mathbf{Q}' respectively, obtained by restricting the object-classes to the volume-preserving quantum families.

Remark 3.2.11. *We shall not need the full strength of the definition of twisted spectral data here; in particular the third condition in the definition 3.2.6. However, we shall continue to work with the usual definition of R -twisted spectral data, keeping in mind that all our results are valid without assuming the third condition.*

Let us now fix a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ which is of compact type. The C^* -algebra generated by \mathcal{A}^∞ in $\mathcal{B}(\mathcal{H})$ will be denoted by \mathcal{A} . Let $\lambda_0 = 0, \lambda_1, \lambda_2, \dots$ be the eigenvalues of D with V_i denoting the (d_i -dimensional, $0 \leq d_i < \infty$) eigenspace for λ_i . Let $\{e_{ij}, j = 1, \dots, d_i\}$ be an orthonormal basis of V_i . We also assume that there is a positive invertible R on \mathcal{H} such that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is an R -twisted spectral data. The operator R must have the form $R|_{V_i} = R_i$, say, with R_i positive invertible $d_i \times d_i$ matrix. Let us denote the CQG $A_{u, d_i}(R_i^T)$ by \mathcal{U}_i , with its canonical unitary representation β_i on $V_i \cong \mathbb{C}^{d_i}$, given by $\beta_i(e_{ij}) = \sum_k e_{ik} \otimes u_{kj}^{R_i^T}$. Let \mathcal{U} be the free product of $\mathcal{U}_i, i = 1, 2, \dots$ and $\beta = *_i \beta_i$ be the corresponding free product representation of \mathcal{U} on \mathcal{H} . We shall also consider the corresponding unitary element $\tilde{\beta}$ in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U})$.

Lemma 3.2.12. *Consider the R -twisted spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ and let (\mathcal{S}, U) be a quantum family of volume and orientation preserving isometries of the given spectral triple. Moreover, assume that the map U is faithful in the sense that there is no proper C^* -subalgebra \mathcal{S}_1 of \mathcal{S} such that \tilde{U} belongs to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{S}_1)$. Then we can find a C^* -isomorphism $\phi : \mathcal{U}/\mathcal{I} \rightarrow \mathcal{S}$ between \mathcal{S} and a quotient of \mathcal{U} by a C^* -ideal \mathcal{I} of \mathcal{U} , such that $U = (\text{id} \otimes \phi) \circ (\text{id} \otimes \Pi_{\mathcal{I}}) \circ \beta$, where $\Pi_{\mathcal{I}}$ denotes the quotient map from \mathcal{U} to \mathcal{U}/\mathcal{I} .*

If, furthermore, there is a compact quantum group structure on \mathcal{S} given by a coproduct Δ such that (\mathcal{S}, Δ, U) is an object in \mathbf{Q}_R' , the ideal \mathcal{I} is a Woronowicz C^ -ideal and the C^* -isomorphism $\phi : \mathcal{U}/\mathcal{I} \rightarrow \mathcal{S}$ is a morphism of compact quantum groups.*

Proof : It is clear that U maps V_i into $V_i \otimes \mathcal{S}$ for each i . Let $v_{kj}^{(i)}$ ($j, k = 1, \dots, d_i$) be the elements of \mathcal{S} such that $U(e_{ij}) = \sum_k e_{ik} \otimes v_{kj}^{(i)}$. Note that $v_i := ((v_{kj}^{(i)}))$ is a unitary in $M_{d_i}(\mathbb{C}) \otimes \mathcal{S}$. Moreover, the $*$ -subalgebra generated by all $\{v_{kj}^{(i)}, i \geq 0, j, k \geq 1\}$ must be dense in \mathcal{S} by the assumption of faithfulness.

Consider the $*$ -homomorphism α_i from the finite dimensional C^* algebra $\mathcal{A}_i \cong M_{d_i}(\mathbb{C})$ generated by the rank one operators $\{|e_{ij} \rangle \langle e_{ik}|, j, k = 1, \dots, d_i\}$ to $\mathcal{A}_i \otimes \mathcal{S}$ given by $\alpha_i(y) = \tilde{U}(y \otimes 1)\tilde{U}^*|_{V_i}$. Clearly, the restriction of the functional τ_R on \mathcal{A}_i is nothing but the functional given by $\text{Tr}(R_i \cdot)$, where Tr denotes the usual trace of matrices. Since α_i preserves this functional by assumption, we get, by the universality of \mathcal{U}_i , a C^* -homomorphism from \mathcal{U}_i to \mathcal{S} sending $u_{kj}^{(i)} \equiv u_{kj}^{R_i^T}$ to $v_{kj}^{(i)}$, and by definition of the free product, this induces a C^* -homomorphism, say Π , from \mathcal{U} onto \mathcal{S} , so that $\mathcal{U}/\mathcal{I} \cong \mathcal{S}$, where $\mathcal{I} := \text{Ker}(\Pi)$.

In case \mathcal{S} has a coproduct Δ making it into a compact quantum group and U is a quantum group representation, it is easy to see that the subalgebra of \mathcal{S} generated by $\{v_{kj}^{(i)}, i \geq 0, j, k = 1, \dots, d_i\}$ is a Hopf algebra, with $\Delta(v_{kj}^{(i)}) = \sum_l v_{kl}^{(i)} \otimes v_{lj}^{(i)}$. From this, it follows that Π is Hopf-algebra morphism, hence \mathcal{I} is a Woronowicz C^* -ideal. \square

Theorem 3.2.13. *For any R -twisted spectral triple of compact type $(\mathcal{A}^\infty, \mathcal{H}, D)$, the category \mathbf{Q}_R of quantum families of volume and orientation preserving isometries has a*

universal (initial) object, say $(\tilde{\mathcal{G}}, U_0)$. Moreover, $\tilde{\mathcal{G}}$ has a coproduct Δ_0 such that $(\tilde{\mathcal{G}}, \Delta_0)$ is a compact quantum group and $(\tilde{\mathcal{G}}, \Delta_0, U_0)$ is a universal object in the category \mathbf{Q}'_R . The representation U_0 is faithful.

Proof : Recall the C^* -algebra \mathcal{U} considered before, along with the representation β and the corresponding unitary $\tilde{\beta}$ belonging to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U})$. For any C^* -ideal \mathcal{I} of \mathcal{U} , we shall denote by $\Pi_{\mathcal{I}}$ the canonical quotient map from \mathcal{U} onto \mathcal{U}/\mathcal{I} , and let $\Gamma_{\mathcal{I}} = (\text{id} \otimes \Pi_{\mathcal{I}}) \circ \beta$. Clearly, $\tilde{\Gamma}_{\mathcal{I}} = (\text{id} \otimes \pi_{\mathcal{I}}) \circ \tilde{\beta}$ is a unitary element of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U}/\mathcal{I})$. Let \mathcal{F} be the collection of all those C^* -ideals \mathcal{I} of \mathcal{U} such that $(\text{id} \otimes \omega) \circ \alpha_{\Gamma_{\mathcal{I}}} \equiv (\text{id} \otimes \omega) \circ \text{ad}_{\tilde{\Gamma}_{\mathcal{I}}}$ maps \mathcal{A}^∞ into \mathcal{A}'' for every state ω (equivalently, every bounded linear functional) on \mathcal{U}/\mathcal{I} . This collection is nonempty, since the trivial one-dimensional C^* -algebra \mathbb{C} gives an object in \mathbf{Q}_R and by Lemma 3.2.12 we do get a member of \mathcal{F} . Now, let \mathcal{I}_0 be the intersection of all ideals in \mathcal{F} . We claim that \mathcal{I}_0 is again a member of \mathcal{F} . Indeed, in the notation of Lemma 1.1.2, it is clear that for a in \mathcal{A}^∞ , $(\text{id} \otimes \phi) \circ \tilde{\Gamma}_{\mathcal{I}_0}(a)$ belongs to \mathcal{A}'' for all ϕ in the convex hull of $\Omega \cup (-\Omega)$. Now, for any state ω on $\mathcal{U}/\mathcal{I}_0$, we can find, by Lemma 1.1.2, a net ω_j in the above convex hull (so in particular $\|\omega_j\| \leq 1$ for all j) such that $\omega(x + \mathcal{I}_0) = \lim_j \omega_j(x + \mathcal{I}_0)$ for all x in $\mathcal{U}/\mathcal{I}_0$.

It follows from Lemma 1.1.8 that $(\text{id} \otimes \omega_j)(X) \rightarrow (\text{id} \otimes \omega)(X)$ (in the strong operator topology) for all X belonging to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U}/\mathcal{I}_0)$. Thus, for a in \mathcal{A}^∞ , $(\text{id} \otimes \omega) \circ \alpha_{\tilde{\Gamma}_{\mathcal{I}_0}}(a)$ is the limit of $(\text{id} \otimes \omega_i) \circ \alpha_{\tilde{\Gamma}_{\mathcal{I}_0}}(a)$ in the strong operator topology, hence belongs to \mathcal{A}'' .

We now show that $(\tilde{\mathcal{G}} := \mathcal{U}/\mathcal{I}_0, \Gamma_{\mathcal{I}_0})$ is a universal object in \mathbf{Q}_R . To see this, consider any object (\mathcal{S}, U) of \mathbf{Q}_R . Without loss of generality we can assume U to be faithful, since otherwise we can replace \mathcal{S} by the C^* -subalgebra generated by the elements $\{v_{kj}^{(i)}\}$ appearing in the proof of Lemma 3.2.12. But by Lemma 3.2.12 we can further assume that \mathcal{S} is isomorphic with \mathcal{U}/\mathcal{I} for some \mathcal{I} belonging to \mathcal{F} . Since $\mathcal{I}_0 \subseteq \mathcal{I}$, we have a C^* -homomorphism from $\mathcal{U}/\mathcal{I}_0$ onto \mathcal{U}/\mathcal{I} , sending $x + \mathcal{I}_0$ to $x + \mathcal{I}$, which is clearly a morphism in the category \mathbf{Q}_R . This is indeed the unique such morphism, since it is uniquely determined on the dense subalgebra generated by $\{u_{kj}^{(i)} + \mathcal{I}_0, i \geq 0, j, k \geq 1\}$ of $\tilde{\mathcal{G}}$.

To construct the coproduct on $\tilde{\mathcal{G}} = \mathcal{U}/\mathcal{I}_0$, we first consider $U^{(2)} : \mathcal{H} \rightarrow \mathcal{H} \otimes \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}}$ given by

$$U^{(2)} = (\Gamma_{\mathcal{I}_0})_{(12)}(\Gamma_{\mathcal{I}_0})_{(13)},$$

where U_{ij} is the usual ‘leg-numbering notation’. It is easy to see that $(\tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}}, U^{(2)})$ is an object in the category \mathbf{Q}_R , so by the universality of $(\tilde{\mathcal{G}}, \Gamma_{\mathcal{I}_0})$, we have a unique unital C^* -homomorphism $\Delta_0 : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}}$ satisfying

$$(\text{id} \otimes \Delta_0)(\Gamma_{\mathcal{I}_0}) = U^{(2)}.$$

Letting both sides act on e_{ij} , we get

$$\sum_l e_{il} \otimes (\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) \left(\sum_k u_{lk}^{(i)} \otimes u_{kj}^{(i)} \right) = \sum_l e_{il} \otimes \Delta_0(\pi_{\mathcal{I}_0}(u_{lj}^{(i)})).$$

Comparing coefficients of e_{il} , and recalling that $\tilde{\Delta}(u_{lj}^{(i)}) = \sum_k u_{lk}^{(i)} \otimes u_{kj}^{(i)}$ (where $\tilde{\Delta}$ denotes the coproduct on \mathcal{U}), we have

$$(\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) \circ \tilde{\Delta} = \Delta_0 \circ \pi_{\mathcal{I}_0} \quad (3.2.1)$$

on the linear span of $\{u_{jk}^{(i)}, i \geq 0, j, k \geq 1\}$, and hence on the whole of \mathcal{U} . This implies that $\tilde{\Delta}$ maps $\mathcal{I}_0 = \text{Ker}(\pi_{\mathcal{I}_0})$ into $\text{Ker}(\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) = (\mathcal{I}_0 \otimes 1 + 1 \otimes \mathcal{I}_0) \subset \mathcal{U} \otimes \mathcal{U}$. In other words, \mathcal{I}_0 is a Hopf C^* -ideal, and hence $\tilde{\mathcal{G}} = \mathcal{U}/\mathcal{I}_0$ has the canonical compact quantum group structure as a quantum subgroup of \mathcal{U} . It is clear from the relation (3.2.1) that Δ_0 coincides with the canonical coproduct of the quantum subgroup $\mathcal{U}/\mathcal{I}_0$ inherited from that of \mathcal{U} . It is also easy to see that the object $(\tilde{\mathcal{G}}, \Delta_0, \Gamma_{\mathcal{I}_0})$ is universal in the category \mathbf{Q}'_R , using the fact that (by Lemma 3.2.12) any compact quantum group (\mathcal{S}, U) acting by volume and orientation preserving isometries on the given spectral triple is isomorphic with a quantum subgroup \mathcal{U}/\mathcal{I} , for some Hopf C^* -ideal \mathcal{I} of \mathcal{U} .

Finally, the faithfulness of U_0 follows from the universality by standard arguments which we briefly sketch. If $\mathcal{G}_1 \subset \tilde{\mathcal{G}}$ is a $*$ -subalgebra of $\tilde{\mathcal{G}}$ such that $\widetilde{U_0}$ is an element of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{G}_1)$, it is easy to see that (\mathcal{G}_1, U_0) is also an object in \mathbf{Q}_R , and by definition of universality of $\tilde{\mathcal{G}}$ it follows that there is a unique morphism, say j , from $\tilde{\mathcal{G}}$ to \mathcal{G}_1 . But the map $i \circ j$ is a morphism from $\tilde{\mathcal{G}}$ to itself, where $i : \mathcal{G}_1 \rightarrow \tilde{\mathcal{G}}$ is the inclusion. Again by universality, we have that $i \circ j = \text{id}_{\tilde{\mathcal{G}}}$, so in particular, i is onto, that is, $\mathcal{G}_1 = \tilde{\mathcal{G}}$. \square

Consider the $*$ -homomorphism $\alpha_0 := \alpha_{U_0}$, where $(\tilde{\mathcal{G}}, U_0)$ is the universal object obtained in the previous theorem. For every state ϕ on $\tilde{\mathcal{G}}$, $(\text{id} \otimes \phi) \circ \alpha_0$ maps \mathcal{A} into \mathcal{A}'' . However, in general α_0 may not be faithful even if U_0 is so, and let \mathcal{G} denote the C^* -subalgebra of $\tilde{\mathcal{G}}$ generated by the elements $\{(f \otimes \text{id}) \circ \alpha_0(a) : f \in \mathcal{A}^*, a \in \mathcal{A}\}$.

Remark 3.2.14. *If the spectral triple is even, then all the proofs above go through with obvious modifications.*

Definition 3.2.15. *We shall call \mathcal{G} the quantum group of orientation-preserving isometries of R -twisted spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ and denote it by $\widetilde{QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D, R)}$ or even simply as $\widetilde{QISO_R^+(D)}$. The quantum group $\tilde{\mathcal{G}}$ is denoted by $\widetilde{QISO_R^+(D)}$.*

If the spectral triple is even, then we will denote \mathcal{G} and $\tilde{\mathcal{G}}$ by $\widetilde{QISO_R^+(D, \gamma)}$ and $\widetilde{QISO_R^+(D, \gamma)}$ respectively.

3.2.3 Stability and topological action

In this subsection, we are going to use the notations as in subsection 3.2.2, in particular $\tilde{\mathcal{G}}, \mathcal{G}, U_0, \alpha_0$. It is not clear from the definition and construction of $QISO_R^+(D)$ whether the C^* algebra \mathcal{A} generated by \mathcal{A}^∞ is stable under α_0 in the sense that $(\text{id} \otimes \phi) \circ \alpha_0$ maps \mathcal{A} into \mathcal{A} for every ϕ . Moreover, even if \mathcal{A} is stable, the question remains whether α_0 is a C^* -action of the CQG $QISO_R^+(D)$. In chapter 5, subsection 5.4.2 we have given an example of a spectral triple for which the $*$ -homomorphism α_0 is not a C^* action. However, one can prove that α_0 is a C^* action for a rather large class of spectral triples, including the cases mentioned below.

(i) For any spectral triple for which there is a ‘reasonable’ Laplacian in the sense of [30]. This includes all classical spectral triples as well as their Rieffel deformation (with $R = I$).

(ii) Under the assumption that there is an eigenvalue of D with a one-dimensional eigenspace spanned by a cyclic separating vector ξ such that any eigenvector of D belongs to the span of $\mathcal{A}^\infty \xi$ and $\{a \in \mathcal{A}^\infty : a\xi \text{ is an eigenvector of } D\}$ is norm-dense in \mathcal{A} (to be proved in subsection 3.2.4).

Now we prove the sufficiency of the condition (i).

We begin with a sufficient condition for stability of \mathcal{A}^∞ under α_0 . Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a (compact type) spectral triple such that

(1) \mathcal{A}^∞ and $\{[D, a], a \in \mathcal{A}^\infty\}$ are contained in the domains of all powers of the derivations $[D, \cdot]$ and $[|D|, \cdot]$.

We will denote by \tilde{T}_t the one parameter group of $*$ -automorphisms on $\mathcal{B}(\mathcal{H})$ given by $\tilde{T}_t(S) = e^{itD} S e^{-itD}$ for all S in $\mathcal{B}(\mathcal{H})$ which is clearly continuous in SOT. We will denote the generator of this group by δ . For X such that $[D, X]$ is bounded, we have $\delta(X) = i[D, X]$ and hence

$$\left\| \tilde{T}_t(X) - X \right\| = \left\| \int_0^t \tilde{T}_s([D, X]) ds \right\| \leq t \| [D, X] \|.$$

Let us say that the spectral triple satisfies the *Sobolev condition* if

$$\mathcal{A}^\infty = \mathcal{A}'' \bigcap_{n \geq 1} \text{Dom}(\delta^n).$$

Then we have the following result, which is a natural generalization of the classical situation, where a measurable isometric action automatically becomes topological (in fact smooth).

Theorem 3.2.16. (i) For every state ϕ on \mathcal{G} ,

$$(\text{id} \otimes \phi) \circ \alpha_0(\mathcal{A}^\infty) \text{ belongs to } \mathcal{A}'' \bigcap_{n \geq 1} \text{Dom}(\delta^n).$$

(ii) If the spectral triple satisfies the Sobolev condition then \mathcal{A}^∞ (and hence \mathcal{A}) is stable under α_0 .

Proof: Since U_0 commutes with $D \otimes I$, it is clear that the automorphism group \tilde{T}_t commutes with $\alpha_0^\phi \equiv (\text{id} \otimes \phi) \circ \alpha_0$, and thus by the continuity of α_0 in the strong operator topology it is easy to see that, for a in $\text{Dom}(\delta)$,

$$\begin{aligned} & \lim_{t \rightarrow 0+} \frac{\tilde{T}_t(\alpha_0^\phi(a)) - \alpha_0^\phi(a)}{t} \\ &= \lim_{t \rightarrow 0+} \alpha_0^\phi\left(\frac{\tilde{T}_t(a) - a}{t}\right) \\ &= \alpha_0^\phi(\delta(a)). \end{aligned}$$

Thus, α_0^ϕ leaves $\text{Dom}(\delta)$ invariant and commutes with δ . Proceeding similarly, we prove (i). The assertion (ii) is a trivial consequence of (i) and the Sobolev condition. \square

Let us now assume

(2) The spectral triple is Θ -summable, that is, for every $t > 0$, e^{-tD^2} is trace-class and the functional $\tau(X) = \text{Lim}_{t \rightarrow 0} \frac{\text{Tr}(Xe^{-tD^2})}{\text{Tr}(e^{-tD^2})}$ (where Lim is as in subsection 1.5.2), is a positive faithful trace on the $*$ algebra, say \mathcal{S}^∞ , generated by $\{\tilde{T}_s(\mathcal{A}^\infty), \tilde{T}_s(\mathcal{A}^\infty)([D, a]) : a \in \mathcal{A}^\infty\}$.

The functional τ is to be interpreted as the volume form (we refer to [29], [30] for the details). The completion of \mathcal{S}^∞ in the norm of $\mathcal{B}(\mathcal{H})$ is denoted by \mathcal{S} , and we shall denote by $\|a\|_2$ and $\|\cdot\|_\infty$ the L^2 -norm $\tau(a^*a)^{\frac{1}{2}}$ and the operator norm of $\mathcal{B}(\mathcal{H})$ respectively.

From the definition of τ , it is also clear that \tilde{T}_t preserves τ , so extends to a group of unitaries on $\mathcal{N} := L^2(\mathcal{S}^\infty, \tau)$. Moreover, for X such that $[D, X]$ is in $\mathcal{B}(\mathcal{H})$, in particular for X in \mathcal{S}^∞ , we have

$$\begin{aligned} & \left\| \tilde{T}_s(X) - X \right\|_2^2 \\ &= \tau(\tilde{T}_s(X)^*(\tilde{T}_s(X) - X)) + \tau(X^*(X - \tilde{T}_s(X))) \\ &\leq 2 \left\| X - \tilde{T}_s(X) \right\|_\infty \|X\|_2 \\ &\leq 2s \| [D, X] \|_\infty \|X\|_2, \end{aligned}$$

which clearly shows that $s \mapsto \tilde{T}_s(X)$ is L^2 -continuous for X belonging to \mathcal{S}^∞ , hence (by unitarity of \tilde{T}_s) on the whole of \mathcal{N} , that is, it is a strongly continuous one-parameter group of unitaries. Let us denote its generator by $\tilde{\delta}$, which is a skew adjoint map, that is, $i\tilde{\delta}$ is self adjoint, and $\tilde{T}_t = \exp(t\tilde{\delta})$. Clearly, $\tilde{\delta} = \delta = [D, \cdot]$ on \mathcal{S}^∞ .

We will denote $L^2(\mathcal{A}^\infty, \tau) \subset \mathcal{N}$ by \mathcal{H}_D^0 and the restriction of $\tilde{\delta}$ to \mathcal{H}_D^0 (which is a closable map from \mathcal{H}_D^0 to \mathcal{N}) by d_D . Thus, d_D is closable too.

We now recall the assumptions made in chapter 2, subsection 2.1.2 for defining the ‘Laplacian’ and the corresponding quantum isometry group of a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$.

The following conditions will also be assumed throughout the rest of this subsection:

- (3) $\mathcal{A}^\infty \subseteq \text{Dom}(\mathcal{L})$ where $\mathcal{L} \equiv \mathcal{L}_D := -d_D^* d_D$.
- (4) \mathcal{L} has compact resolvent.
- (5) Each eigenvector of \mathcal{L} (which has a discrete spectrum , hence a complete set of eigenvectors) belongs to \mathcal{A}^∞ .
- (6) The complex linear span of the eigenvectors of \mathcal{L} , say \mathcal{A}_0^∞ (which is a subspace of \mathcal{A}^∞ by assumption (5)), is norm dense in \mathcal{A}^∞ .

It is clear that \mathcal{L} maps (\mathcal{A}_0^∞) into itself. The $*$ -subalgebra of \mathcal{A}^∞ generated by \mathcal{A}_0^∞ is denoted by \mathcal{A}_0 . We also note that $\mathcal{L} = P_0 \tilde{\mathcal{L}} P_0$, where $\tilde{\mathcal{L}} := (i\tilde{\delta})^2$ (which is a self adjoint operator on \mathcal{N}) and P_0 denotes the orthogonal projection in \mathcal{N} whose range is the subspace \mathcal{H}_D^0 .

Theorem 3.2.17. *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple satisfying the assumptions (1) – (6) made above. In addition, assume that at least one of conditions (a) and (b) mentioned below is satisfied:*

- (a) $\mathcal{A}'' \subseteq \mathcal{H}_D^0$.
- (b) $\alpha_0^\phi(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$ for every state ϕ on $\mathcal{G} = QISO_I^+(D)$.

Then α_0 is a C^ -action of $QISO_I^+(D)$ on \mathcal{A} .*

Proof : Under either of the conditions (a) and (b), for any fixed ϕ , the map α_0^ϕ maps \mathcal{A}^∞ into the subset \mathcal{H}_D^0 of \mathcal{N} . Since α_0^ϕ also commutes with $[D, \cdot]$ on \mathcal{A}^∞ , it is clear that α_0^ϕ maps \mathcal{S}^∞ into \mathcal{N} . In fact, using the complete positivity of the map α_0^ϕ and the α_0 -invariance of τ , we see that

$$\tau(\alpha_0^\phi(a)^* \alpha_0^\phi(a)) \leq \tau(\alpha_0^\phi(a^* a)) = (\text{id} \otimes \phi)((\tau \otimes \text{id})\alpha_0(a^* a)) = \tau(a^* a).1$$

which implies that α_0^ϕ extends to a bounded operator from \mathcal{N} to itself. Since U_0 commutes with D , it is clear that α_0^ϕ (viewed as a bounded operator on \mathcal{N}) will commute

with the group of unitaries \tilde{T}_t , hence with its generator $\tilde{\delta}$ and also with the self adjoint operator $\tilde{\mathcal{L}} = (i\tilde{\delta})^2$.

On the other hand, it follows from the definition of $\mathcal{G} = QISO_I^+(D)$ that $(\tau \otimes \text{id})(\alpha_0(X)) = \tau(X)1_{\mathcal{G}}$ for all X in $\mathcal{B}(\mathcal{H})$, in particular for X belonging to \mathcal{S}^∞ , and thus the map $\mathcal{S}^\infty \otimes_{\text{alg}} \mathcal{G} \ni (a \otimes q) \mapsto \alpha_0(a)(1 \otimes q)$ extends to a \mathcal{G} -linear unitary, denoted by W (say), on the Hilbert \mathcal{G} -module $\mathcal{N} \otimes \mathcal{G}$. Note that here we have used the fact (which that for any ϕ , $(\text{id} \otimes \phi)(W)(\mathcal{S}^\infty \otimes_{\text{alg}} \mathcal{G}) \subseteq \mathcal{N}$, since $\alpha_0^\phi(\mathcal{S}^\infty) \subseteq \mathcal{N}$). The commutativity of α_0^ϕ with \tilde{T}_t for every ϕ clearly implies that W and $\tilde{T}_t \otimes \text{id}_{\mathcal{G}}$ commute on $\mathcal{N} \otimes \mathcal{G}$. Moreover, α_0^ϕ maps \mathcal{H}_D^0 into itself, so W maps $\mathcal{H}_D^0 \otimes \mathcal{G}$ into itself, and hence (by unitarity of W) it commutes with the projection $P_0 \otimes 1$. It follows that α_0^ϕ commutes with P_0 , and (since it also commutes with $\tilde{\mathcal{L}}$) commutes with $\mathcal{L} = P_0 \tilde{\mathcal{L}} P_0$ as well.

Thus, α_0^ϕ preserves each of the (finite dimensional) eigenspaces of the Laplacian \mathcal{L} , and so is a Hopf algebraic action on the subalgebra \mathcal{A}_0 spanned algebraically by these eigenvectors. Moreover, the \mathcal{G} -linear unitary W clearly restricts to a unitary representation on each of the above eigenspaces. If we denote by $((q_{ij}))_{(i,j)}$ the \mathcal{G} -valued unitary matrix corresponding to one such particular eigenspace, then by Proposition 1.2.23, q_{ij} must belong to \mathcal{G}_0 and we must have $\epsilon(q_{ij}) = \delta_{ij}$ (Kronecker delta). This implies $(\text{id} \otimes \epsilon) \circ \alpha_0 = \text{id}$ on each of the eigenspaces, hence on the norm-dense subalgebra \mathcal{A}_0 of \mathcal{A} , completing the proof of the fact that α_0 extends to a C^* action on \mathcal{A} . \square

Combining the above theorem with Theorem 3.2.16, we get the following immediate corollary.

Corollary 3.2.18. *If the spectral triple satisfies the Sobolev condition mentioned before, in addition to the assumptions 1 – 6, then $QISO_I^+(D)$ has a C^* -action. In particular, for a classical spectral triple, $QISO_I^+(D)$ has C^* -action.*

Remark 3.2.19. *Let us remark here that in case the restriction of τ on \mathcal{A}^∞ is normal, that is, continuous with respect to the weak operator topology inherited from $\mathcal{B}(\mathcal{H})$, then \mathcal{H}_D^0 will contain \mathcal{A}'' , which is the closure of \mathcal{A}^∞ in the weak operator topology of $\mathcal{B}(\mathcal{H})$, so the condition (a) of Theorem 3.2.17 (and hence its conclusion) holds.*

Remark 3.2.20. *In a private communication to us, Shuzhou Wang has kindly pointed out that a possible alternative approach to the formulation of quantum group of isometries may involve the category of CQG which has a C^* -action on the underlying C^* algebra and a unitary representation with respect to which the Dirac operator is equivariant. However, we see from Corollary 5.4.17 of chapter 5 that the category proposed by Wang does not admit a universal object in general.*

3.2.4 Universal object in the categories \mathbf{Q} or \mathbf{Q}'

We shall now investigate further conditions on the spectral triple which will ensure the existence of a universal object in the category \mathbf{Q} or \mathbf{Q}' . Whenever such a universal object exists we shall denote it by $\widetilde{QISO}^+(D)$, and denote by $QISO^+(D)$ its largest Woronowicz subalgebra for which α_U on \mathcal{A}^∞ (where U is the unitary representation of $\widetilde{QISO}^+(D)$ on \mathcal{H}) is faithful.

Remark 3.2.21. *If $\widetilde{QISO}^+(D)$ exists, then by Proposition 3.2.7, there will exist some R such that $\widetilde{QISO}^+(D)$ is an object in the category $\mathbf{Q}'_R(D)$. Since the universal object in this category, that is, $\widetilde{QISO}_R^+(D)$, is clearly a sub-object of $\widetilde{QISO}^+(D)$, we have $\widetilde{QISO}^+(D) \cong \widetilde{QISO}_R^+(D)$ for this choice of R .*

Let us state and prove a result below, which gives some sufficient conditions for the existence of $\widetilde{QISO}^+(D)$.

Theorem 3.2.22. *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple of compact type as before and assume that D has an one-dimensional eigenspace spanned by a unit vector ξ , which is cyclic and separating for the algebra \mathcal{A}^∞ . Moreover, assume that each eigenvector of D belongs to the dense subspace $\mathcal{A}^\infty \xi$ of \mathcal{H} . Then there is a universal object, $(\widetilde{\mathcal{G}}, U_0)$. Moreover, $\widetilde{\mathcal{G}}$ has a coproduct Δ_0 such that $(\widetilde{\mathcal{G}}, \Delta_0)$ is a compact quantum group and $(\widetilde{\mathcal{G}}, \Delta_0, U_0)$ is a universal object in the category \mathbf{Q}' .*

If we denote by \mathcal{G} the Woronowicz C^ subalgebra of $\widetilde{\mathcal{G}}$ generated by elements of the form $\langle \alpha_{U_0}(a)(\eta \otimes 1), \eta' \otimes 1 \rangle_{\widetilde{\mathcal{G}}}$ where η, η' are in \mathcal{H} , a in \mathcal{A}^∞ and $\langle \cdot, \cdot \rangle_{\widetilde{\mathcal{G}}}$ denotes the $\widetilde{\mathcal{G}}$ valued inner product of $\mathcal{H} \otimes \widetilde{\mathcal{G}}$, we have $\widetilde{\mathcal{G}} \cong \mathcal{G} * C(\mathbb{T})$.*

Proof: Let $V_i, \{e_{ij}\}$ be as before, and by assumption we have $e_{ij} = x_{ij}\xi$ for a unique x_{ij} in \mathcal{A}^∞ . Clearly, since ξ is separating, the vectors $\{\overline{e_{ij}} = x_{ij}^*\xi, j = 1, \dots, d_i\}$ are linearly independent, so the matrix $Q_i = ((\langle \overline{e_{ij}}, \overline{e_{ik}} \rangle))_{j,k=1}^{d_i}$ is positive and invertible. Now, given a quantum family of orientation-preserving isometries (\mathcal{S}, U) , we must have $\widetilde{U}(\xi \otimes 1) = \xi \otimes q$, say, for some q in \mathcal{S} , and from the unitarity of \widetilde{U} it follows that q is a unitary element. Moreover, U leaves V_i invariant, so let $\widetilde{U}(e_{ij} \otimes 1) = \sum_k e_{ik} \otimes v_{kj}^{(i)}$. But this can be rewritten as

$$\alpha_U(x_{ij})(\xi \otimes q) = \sum_k x_{ik}\xi \otimes v_{kj}^{(i)}.$$

Since ξ is separating and q is unitary, this implies $\alpha_U(x_{ij}) = \sum_k x_{ik} \otimes v_{kj}^{(i)} q^*$, and thus we have

$$\widetilde{U}(\overline{e_{ij}} \otimes 1) = \alpha_U(x_{ij})^*(\xi \otimes q) = \sum_k x_{ik}^* \xi \otimes q(v_{kj}^{(i)})^* q = \sum_k \overline{e_{ik}} \otimes q(v_{kj}^{(i)})^* q.$$

Taking the \mathcal{S} -valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ on both sides of the above expression, and using the fact that U preserves this \mathcal{S} -valued inner product, we obtain

$$\begin{aligned}
& \langle \overline{e_{ij}}, \overline{e_{ij'}} \rangle . 1 \\
&= \left\langle \tilde{U}(\overline{e_{ij}} \otimes 1), \tilde{U}(\overline{e_{ij'}} \otimes 1) \right\rangle \\
&= \left\langle \sum_k \overline{e_{ik}} \otimes q(v_{kj}^{(i)})^* q, \sum_{k'} \overline{e_{ik'}} \otimes q(v_{k'j'}^{(i)})^* q \right\rangle \\
&= \sum_{k,k'} \langle \overline{e_{ik}}, \overline{e_{ik'}} \rangle q^* v_{kj}^{(i)} q^* q(v_{k'j'}^{(i)})^* q.
\end{aligned}$$

This implies,

$$\begin{aligned}
& q(Q_i)_{jj'} q^* . 1 \\
&= \sum_{k,k'} v_{kj}^{(i)} \langle \overline{e_{ik}}, \overline{e_{ik'}} \rangle (v_{k'j'}^{(i)})^* .
\end{aligned}$$

Thus,

$$(Q_i)_{jj'} = \sum_{k,k'} v_{kj}^{(i)} (Q_i)_{kk'} (v_{k'j'}^{(i)})^* .$$

Hence, we have $Q_i = v'_i Q_i \overline{v_i}$ (where $v_i = ((v_{kj}^{(i)}))$). Thus, $Q_i^{-1} v'_i Q_i$ must be the (both-sided) inverse of $\overline{v_i}$. Thus, we get a canonical surjective morphism from $A_{u,d_i}(Q_i)$ to the C^* algebra generated by $\{v_{kj}^{(i)} : j, k = 1, 2, \dots, d_i\}$. This induces a surjective morphism from the free product of $A_{u,d_i}(Q_i)$, $i = 1, 2, \dots$ onto \mathcal{S} . The rest of the arguments for showing the existence of $\tilde{\mathcal{G}}$ will be quite similar to the arguments used in the proof of Theorem 3.2.13, hence omitted.

Now we come to the proof of the last part of the theorem. For a in \mathcal{A}^∞ , $\tilde{U}(a\xi \otimes 1) = \alpha_U(a)\tilde{U}(\xi \otimes 1) = \alpha_U(a)(\xi \otimes q)$. Now, recalling that $\text{Span}\{a\xi : a \in \mathcal{A}^\infty\}$ is dense in \mathcal{H} , it is clear that $\tilde{\mathcal{G}} = \mathcal{G} * C^*(q) \cong \mathcal{G} * C(\mathbb{T})$. \square

Remark 3.2.23. *Some of the examples considered in section 3.4 will show that the conditions of the above theorem are not actually necessary; $\widehat{QISO}^+(D)$ may exist without the existence of a single cyclic separating eigenvector as above.*

Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple of compact type satisfying the conditions of the above theorem. Let the faithful vector state corresponding to the cyclic separating vector ξ be denoted by τ . Let $\mathcal{A}_{00} = \text{span}\{a \in \mathcal{A}^\infty : a\xi \text{ is an eigenvector of } D\}$.

Moreover, assume that \mathcal{A}_{00} is norm dense in \mathcal{A}^∞ .

Let $\hat{D} : \mathcal{A}_{00} \rightarrow \mathcal{A}_{00}$ be defined by :

$$\hat{D}(a)\xi = D(a\xi).$$

This is well defined since ξ is cyclic and separating.

Definition 3.2.24. Let \mathcal{A} be a C^* algebra and \mathcal{A}^∞ be a dense $*$ -subalgebra. Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple of compact type as above.

Let $\widehat{\mathcal{C}}$ be the category with objects (\mathcal{Q}, α) such that \mathcal{Q} is a compact quantum group with a C^* action α on \mathcal{A} such that:

1. α is τ preserving (where τ is as above), that is, $(\tau \otimes \text{id})(\alpha(a)) = \tau(a).1$.
2. α maps \mathcal{A}_{00} inside $\mathcal{A}_{00} \otimes_{\text{alg}} \mathcal{Q}$.
3. $\alpha \widehat{D} = (\widehat{D} \otimes I)\alpha$.

Corollary 3.2.25. There exists a universal object $\widehat{\mathcal{Q}}$ of the category $\widehat{\mathcal{C}}$ and it is isomorphic to the Woronowicz C^* subalgebra $\mathcal{G} = QISO^+(D)$ of $\widetilde{\mathcal{G}}$ obtained in Theorem 3.2.22.

Proof : The proof of the existence of the universal object follows verbatim from the proof of Theorem 2.1.5 replacing \mathcal{L} by \widehat{D} and noting that D has compact resolvent. We denote by $\widehat{\alpha}$ the action of $\widehat{\mathcal{Q}}$ on \mathcal{A} .

Now, we prove that $\widehat{\mathcal{Q}}$ is isomorphic to \mathcal{G} .

Each eigenvector of D is in \mathcal{A}^∞ by assumption. It is easily observed from the proof of Theorem 3.2.22 that α_{U_0} maps the norm-dense $*$ -subalgebra \mathcal{A}_{00} into $\mathcal{A}_{00} \otimes_{\text{alg}} \mathcal{G}_0$, and $(\text{id} \otimes \epsilon) \circ \alpha_{U_0} = \text{id}$, so that α_{U_0} is indeed a C^* action of the CQG \mathcal{G} . Moreover, it can be easily seen that τ preserves α_{U_0} and that α_{U_0} commutes with \widehat{D} . Therefore, $(\mathcal{G}, \alpha_{U_0})$ is an element of $\text{Obj}(\widehat{\mathcal{C}})$ and hence \mathcal{G} is a quantum subgroup of $\widehat{\mathcal{Q}}$ by the universality of $\widehat{\mathcal{Q}}$.

For the converse, we start by showing that $\widehat{\alpha}$ induces a unitary representation W of $\widehat{\mathcal{Q}} * C(\mathbb{T})$ on \mathcal{H} which commutes with D , and the corresponding conjugated action α_W coincides with $\widehat{\alpha}$.

Define $W(a\xi) = \widehat{\alpha}(a)(\xi)(1 \otimes q^*)$ for all a in \mathcal{A}_0^∞ where q is a generator of $C(\mathbb{T})$.

Since we have $(\tau \otimes \text{id})(\alpha(a)) = \tau(a).1$, it follows that \widetilde{W} is a $(\widehat{\mathcal{Q}} * C(\mathbb{T}))$ -linear isometry on the dense subspace $\mathcal{A}_{00}\xi \otimes_{\text{alg}} \widehat{\mathcal{Q}}$ and thus extends to $\mathcal{H} \otimes (\widehat{\mathcal{Q}} * C(\mathbb{T}))$ as an isometry. Moreover, since $\widehat{\alpha}(\mathcal{A})(1 \otimes \widehat{\mathcal{Q}})$ is norm dense in $\mathcal{A} \otimes \widehat{\mathcal{Q}}$ (by the definition of a CQG action) it is clear that the range of \widetilde{W} is dense, so \widetilde{W} is indeed a unitary. It is quite obvious that it is a unitary representation of $\widehat{\mathcal{Q}} * C(\mathbb{T})$.

We also have,

$$\begin{aligned} WD(a\xi) &= W(\widehat{D}(a)\xi) = \widehat{\alpha}(\widehat{D}(a))(\xi)(1 \otimes q^*) \\ &= (D \otimes I)(\widehat{\alpha}(a)\xi)(1 \otimes q^*) = (D \otimes I)W(a\xi), \end{aligned}$$

that is, W commutes with D .

Moreover, it is easy to observe that $\alpha_W = \widehat{\alpha}$. This gives a surjective CQG morphism from $\widetilde{\mathcal{G}} = \mathcal{G} * C(\mathbb{T})$ to $\widehat{\mathcal{Q}} * C(\mathbb{T})$, sending \mathcal{G} onto $\widehat{\mathcal{Q}}$, which completes the proof. \square

3.3 Comparison with the approach of [30] based on Laplacian

Throughout this section, we shall assume the set-up of subsection 3.2.3 for the existence of a ‘Laplacian’, including assumptions **1** – **6**. Let us also use the notation of that subsection.

We recall from chapter 2 that a CQG (\mathcal{S}, Δ) which has an action α on \mathcal{A} is said to act smoothly and isometrically on the noncommutative manifold (of compact type) $(\mathcal{A}^\infty, \mathcal{H}, D)$ if $(\text{id} \otimes \phi) \circ \alpha(\mathcal{A}_0^\infty) \subseteq \mathcal{A}_0^\infty$ for every state ϕ on \mathcal{S} , and also $(\text{id} \otimes \phi) \circ \alpha$ commutes with the Laplacian $\mathcal{L} \equiv \mathcal{L}_D$ on \mathcal{A}_0^∞ (where \mathcal{A}_0^∞ is the complex linear span of the eigenvectors of \mathcal{L}). One can consider the category $\mathbf{Q}'_{\mathcal{L}_D}$ of all compact quantum groups acting smoothly and isometrically on \mathcal{A} , where the morphisms are quantum group morphisms which intertwine the actions on \mathcal{A} . We make the following additional assumption throughout the present section:

(7) There exists a universal object in $\mathbf{Q}'_{\mathcal{L}_D}$ (the quantum isometry group for the Laplacian $\mathcal{L} \equiv \mathcal{L}_D$ in the sense of [30]), and it is denoted by $QISO^{\mathcal{L}} \equiv QISO^{\mathcal{L}_D}$

The following result now follows immediately from Theorem 3.2.17 of subsection 2.3.

Corollary 3.3.1. *If $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a spectral triple (of compact type) satisfying any of the two conditions (a) or (b) of Theorem 3.2.17, then $QISO_I^+(D)$ is a sub-object of $QISO^{\mathcal{L}_D}$ in the category $\mathbf{Q}'_{\mathcal{L}_D}$.*

Proof: The proof is a consequence of the fact that $QISO_I^+(D)$ has the C^* -action α_0 on \mathcal{A} , and the observation already made in the proof of the Theorem 3.2.17 that this action commutes with the Laplacian \mathcal{L}_D . \square

Now, we will need the Hilbert space of forms \mathcal{H}_{d+d^*} corresponding to a Θ -summable spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ as discussed in subsection 1.5.2. We recall that one obtains an associated spectral triple $(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d+d^*)$. We assume that this spectral triple is of compact type, that is, $d+d^*$ has compact resolvents.

We will denote the inner product on the space of k forms coming from the spectral triples $(\mathcal{A}^\infty, \mathcal{H}, D)$ and $(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d+d^*)$ by $\langle \cdot, \cdot \rangle_{\mathcal{H}_D^k}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_{d+d^*}^k}$ respectively, $k = 0, 1$.

We will denote by π_D, π_{d+d^*} the representations of \mathcal{A}^∞ in \mathcal{H} and \mathcal{H}_{d+d^*} respectively.

Let U_{d+d^*} be the canonical unitary representation of $QISO_I^+(d+d^*)$ on \mathcal{H}_{d+d^*} .

\mathcal{H}_{d+d^*} breaks up into finite dimensional orthogonal subspaces corresponding to the distinct eigenvalues of $\Delta := (d+d^*)^2 = d^*d + dd^*$. It is easy to see that Δ leaves each of the subspaces \mathcal{H}_D^i invariant, and we will denote by $V_{\lambda,i}$ the subspace of $\mathcal{H}_{d+d^*}^i$ spanned

by eigenvectors of Δ corresponding to the eigenvalue λ . Let $\{e_{j,\lambda,i}\}_j$ be an orthonormal basis of $V_{\lambda,i}$. Note that \mathcal{L}_D is the restriction of Δ to \mathcal{H}_D^0 .

Now we recall Proposition 2.1.8. It was shown there that $QISO^{\mathcal{L}_D}$ has a unitary representation $U \equiv U_{\mathcal{L}}$ on \mathcal{H}^{d+d^*} such that U commutes with $d + d^*$. Thus, $(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d + d^*)$ is a $QISO^{\mathcal{L}_D}$ equivariant spectral triple. Moreover, by Remark 2.1.6, $QISO^{\mathcal{L}_D}$ has tracial Haar state, which implies, by Proposition 3.2.7 and Remark 3.2.8 that α_U keeps the functional τ_I invariant. Summarizing, we have the following result:

Proposition 3.3.2. *The quantum isometry group $(QISO^{\mathcal{L}_D}, U_{\mathcal{L}})$ is a sub-object of $(QISO_I^+(d + d^*), U_{d+d^*})$ in the category $\mathbf{Q}_I(d + d^*)$, so in particular, $QISO^{\mathcal{L}_D}$ is isomorphic to a quotient of $QISO_I^+(d + d^*)$ by a Woronowicz C^* ideal.*

We shall give (under mild conditions) a concrete description of the above Woronowicz ideal.

Let \mathcal{I} be the C^* ideal of $QISO_I^+(d + d^*)$ generated by

$$\cup_{\lambda \in \sigma(\Delta)} \{ \langle (P_0^\perp \otimes \text{id}) U_{d+d^*}(e_{j\lambda 0}), e_{j\lambda i'} \otimes 1 \rangle : j, i' \geq 1 \},$$

where P_0 is the projection onto \mathcal{H}_D^0 , $\langle \cdot, \cdot \rangle$ denotes the $QISO_I^+(d + d^*)$ valued inner product and $\sigma(\Delta)$ denotes the spectrum of Δ .

Since U_{d+d^*} keeps the eigenspaces of $\Delta = (d + d^*)^2$ invariant, we can write

$$U_{d+d^*}(e_{j\lambda 0}) = \sum_k e_{k\lambda 0} \otimes q_{kj\lambda 0} + \sum_{i' \neq 0, k'} e_{k'\lambda i'} \otimes q_{k'j\lambda i'},$$

for some $q_{kj\lambda 0}, q_{k'j\lambda i'}$ in $QISO_I^+(d + d^*)$.

We note that $q_{k'j\lambda i'}$ is in \mathcal{I} if $i' \neq 0$.

Lemma 3.3.3. *\mathcal{I} is a co-ideal of $QISO_I^+(d + d^*)$.*

Proof: It is enough to prove the relation $\Delta(X) \in \mathcal{I} \otimes QISO_I^+(d + d^*) + QISO_I^+(d + d^*) \otimes \mathcal{I}$ for the elements X in \mathcal{I} of the form $\langle (P_0^\perp \otimes \text{id}) U_{d+d^*}(e_{j\lambda 0}), e_{j\lambda i_0} \otimes 1 \rangle$. We have:

$$\begin{aligned} & \Delta(\langle (P_0^\perp \otimes \text{id}) U_{d+d^*}(e_{m\lambda 0}), e_{j\lambda i_0} \otimes 1 \rangle) \\ &= \langle (P_0^\perp \otimes \text{id})(\text{id} \otimes \Delta) U_{d+d^*}(e_{m\lambda 0}), e_{j\lambda i_0} \otimes 1 \otimes 1 \rangle \\ &= \langle (P_0^\perp \otimes \text{id}) U_{(12)} U_{(13)}(e_{m\lambda 0}), e_{j\lambda i_0} \otimes 1 \otimes 1 \rangle \end{aligned}$$

$$\begin{aligned}
&= \left\langle (P_0^\perp \otimes \text{id}) U_{(12)} \left(\sum_k e_{k\lambda 0} \otimes 1 \otimes q_{km\lambda 0} \right) , e_{j\lambda i_0} \otimes 1 \otimes 1 \right\rangle \\
&+ \sum_{i' \neq 0, l} \left\langle (P_0^\perp \otimes \text{id}) U_{(12)} (e_{l\lambda i'} \otimes 1 \otimes q_{lm\lambda i'}) , e_{j\lambda i_0} \otimes 1 \otimes 1 \right\rangle \\
&= \sum_{k, k'} \left\langle (P_0^\perp \otimes \text{id}) (e_{k'\lambda 0} \otimes q_{k'k\lambda 0} \otimes q_{km\lambda 0}) , e_{j\lambda i_0} \otimes 1 \otimes 1 \right\rangle \\
&+ \sum_{i' \neq 0, k, k''} \left\langle (P_0^\perp \otimes \text{id}) (e_{k''k\lambda i'} \otimes q_{k'',k,\lambda,i'} \otimes q_{km\lambda 0}) , e_{j\lambda i_0} \otimes 1 \otimes 1 \right\rangle \\
&+ \sum_{i' \neq 0, l, l'} \left\langle (P_0^\perp \otimes \text{id}) (e_{l'\lambda i'} \otimes q_{l'l\lambda i'} \otimes q_{lm\lambda i'}) , e_{j\lambda i_0} \otimes 1 \otimes 1 \right\rangle \\
&+ \sum_{i' \neq 0, i'' \neq i', l, l''} \left\langle (P_0^\perp \otimes \text{id}) (e_{l''\lambda i''} \otimes q_{l''l\lambda i''} \otimes q_{lm\lambda i'}) , e_{j\lambda i_0} \otimes 1 \otimes 1 \right\rangle \\
&= \sum_{i' \neq 0, k', k''} \langle e_{k''\lambda i'} \otimes q_{k''k\lambda i'} \otimes q_{km\lambda 0} , e_{j\lambda i_0} \otimes 1 \otimes 1 \rangle \\
&+ \sum_{i' \neq 0, l, l'} \langle e_{l'\lambda i'} \otimes q_{l'l\lambda i'} \otimes q_{lm\lambda i'} , e_{j\lambda i_0} \otimes 1 \otimes 1 \rangle \\
&+ \sum_{i' \neq 0, i'' \neq i', i'' \neq 0, l, l''} \langle e_{l''\lambda i''} \otimes q_{l''l\lambda i''} \otimes q_{lm\lambda i'} , e_{j\lambda i_0} \otimes 1 \otimes 1 \rangle ,
\end{aligned}$$

which is clearly in $\mathcal{I} \otimes QISO_I^+(d + d^*) + QISO_I^+(d + d^*) \otimes \mathcal{I}$, as $q_{kj\lambda i'}$ is an element of \mathcal{I} for $i' \neq 0$. \square

Theorem 3.3.4. *If $\alpha_{U_{d+d^*}}$ is a C^* action on \mathcal{A} , then we have $QISO^{\mathcal{L}_D} \cong QISO_I^+(d + d^*)/\mathcal{I}$.*

Proof : By Proposition 3.3.2, we conclude that there exists a surjective CQG morphism $\pi : QISO_I^+(d + d^*) \rightarrow QISO^{\mathcal{L}_D}$. By construction (as in Proposition 2.1.8), the unitary representation $U_{\mathcal{L}}$ of $QISO^{\mathcal{L}_D}$ preserves each of the \mathcal{H}_D^i , in particular \mathcal{H}_D^0 . It is

then clear from the definition of \mathcal{I} that π induces a surjective CQG morphism (in fact, a morphism in the category $\mathbf{Q}'_I(d + d^*)$) $\pi' : QISO_I^+(d + d^*)/\mathcal{I} \rightarrow QISO^{\mathcal{L}_D}$.

Conversely, if $V = (\text{id} \otimes \rho_{\mathcal{I}}) \circ U_{d+d^*}$ is the representation of $QISO_I^+(d + d^*)/\mathcal{I}$ on \mathcal{H}_{d+d^*} induced by U_{d+d^*} (where $\rho_{\mathcal{I}} : QISO_I^+(d + d^*) \rightarrow QISO_I^+(d + d^*)/\mathcal{I}$ denotes the quotient map), then V preserves \mathcal{H}_D^0 (by definition of \mathcal{I}), so commutes with P_0 . Since V also commutes with $(d + d^*)^2$, it follows that V must commute with $(d + d^*)^2 P_0 = \mathcal{L}$, that is,

$$\tilde{V}(d^* d P_0 \otimes 1) = (d^* d P_0 \otimes 1) \tilde{V}.$$

It is easy to show from the above that α_V (which is a C^* action on \mathcal{A} since $\alpha_{U_{d+d^*}}$ is so by assumption) is a smooth isometric action of $QISO_I^+(d + d^*)/\mathcal{I}$ in the sense of [30], with respect to the Laplacian \mathcal{L} . This implies that $QISO_I^+(d + d^*)/\mathcal{I}$ is a sub-object of $QISO^{\mathcal{L}_D}$ in the category $\mathbf{Q}'_{\mathcal{L}_D}$, and completes the proof. \square

Now we prove that under some further assumptions which are valid for classical manifolds as well as their Rieffel deformation, one even has the isomorphism $QISO^{\mathcal{L}_D} \cong QISO_I^+(d + d^*)$.

We assume the following:

(A) Both the spectral triples $(\mathcal{A}^\infty, \mathcal{H}, D)$ and $(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d + d^*)$ satisfy the assumptions (1) – (7), so in particular both $QISO^{\mathcal{L}_D}$ and $QISO^{\mathcal{L}_{D'}}$ exist (here $D' = d + d^*$).

(B) For all a, b in \mathcal{A}^∞ , we have

$$\langle a, b \rangle_{\mathcal{H}_D^0} = \langle a, b \rangle_{\mathcal{H}_{D'}^0}, \quad \langle d_D a, d_D b \rangle_{\mathcal{H}_D^1} = \langle d_{D'} a, d_{D'} b \rangle_{\mathcal{H}_{D'}^1}.$$

Remark 3.3.5. For classical compact spin manifolds these assumptions can be verified by comparing the local expressions of D^2 and the ‘Hodge Laplacian’ $(D')^2$ in suitable coordinate charts. In fact, in this case, both these operators turn out to be essentially same, upto a ‘first order term’, which is relatively compact with respect to D^2 or $(D')^2$.

By assumption (B), we observe that the identity map on \mathcal{A}^∞ extends to a unitary, say Σ , from \mathcal{H}_D^0 to $\mathcal{H}_{D'}^0$. Moreover, we have

$$\mathcal{L}_D = \Sigma^* \mathcal{L}_{D'} \Sigma,$$

from which we conclude the following:

Proposition 3.3.6. Under the above assumptions, $QISO^{\mathcal{L}_D} \cong QISO^{\mathcal{L}_{D'}}$.

We conclude this section with the following result, which identifies the quantum isometry group $QISO^{\mathcal{L}_D}$ of [30] as the $QISO_I^+$ of a spectral triple, and thus, in some sense, accommodates the construction of [30] in the framework of the present article.

Theorem 3.3.7. *If in addition to the assumptions already made, the spectral triple (of compact type) $(\mathcal{A}^\infty, \mathcal{H}_{D'}, D')$ also satisfies the conditions of Theorem 3.2.17, so that $QISO_I^+(D')$ has a C^* -action, then we have the following isomorphism of CQG s:*

$$QISO^{\mathcal{L}_D} \cong QISO_I^+(D') \cong QISO^{\mathcal{L}_{D'}}.$$

Proof : By Proposition 3.3.2 we have that $QISO^{\mathcal{L}_D}$ is a sub-object of $QISO_I^+(D')$ in the category $\mathbf{Q}'_I(D')$. On the other hand, by Theorem 3.2.17 we have $QISO_I^+(D')$ as a sub-object of $QISO^{\mathcal{L}_{D'}}$ in the category $\mathbf{Q}'_{\mathcal{L}_{D'}}$. Combining these facts with the conclusion of Proposition 3.3.6, we get the required isomorphism. \square

Remark 3.3.8. *The assumptions, and hence the conclusions, of this section are valid also for spectral triples obtained by Rieffel deformation of a classical spectral triple, to be discussed in details in chapter 4.*

3.4 Examples and computations

In this section we compute the quantum group of orientation preserving isometries for spectral triples on $SU_\mu(2)$ and $C(\mathbb{T}^2)$. The computations for the Podles' spheres and Rieffel deformed manifolds are given in chapter 5 and Chapter 4 respectively.

3.4.1 Equivariant spectral triple on $SU_\mu(2)$

We recall from subsection 1.2.4 that by $t_{i,j}^n$ s, we will denote the (i, j) th matrix element of the $(2n+1)$ dimensional representation of $SU_\mu(2)$ and $e_{i,j}^n$ s will denote the normalized (with respect to the Haar state h) $t_{i,j}^n$ s. We consider the spectral triple on $SU_\mu(2)$ constructed by Chakraborty and Pal ([13]) and also discussed thoroughly in [18] which is defined by $(\mathcal{A}^\infty, \mathcal{H}, D)$ where \mathcal{A}^∞ is the linear span of t_{ij}^n s, $\mathcal{H} = L^2(SU_\mu(2))$ and D is defined by :

$$\begin{aligned} D(e_{ij}^n) &= (2n+1)e_{ij}^n, \quad n \neq i \\ &= -(2n+1)e_{ij}^n, \quad n = i. \end{aligned}$$

Here, we have a cyclic separating vector $1_{SU_\mu(2)}$, and the corresponding faithful state is the Haar state h . Thus, we are in the set up of the subsection 3.2.4, and as $\xi = 1$, $\mathcal{A}_{00} = \mathcal{A}^\infty$ in this case. Therefore, an operator commuting with D (equivalently with \widehat{D}) must keep $V_i^l := \text{Span}\{t_{ij}^l : j = -l, \dots, l\}$ invariant for all fixed l and i where \widehat{D} is the operator as in subsection 3.2.4.

In the notation of Corollary 3.2.25, we have $\mathcal{A}_{00} = \text{Span}\{t_{i,j}^l : l = 0, 1/2, \dots\} = \mathcal{A}^\infty$ in this case. All the conditions of Theorem 3.2.22 and Corollary 3.2.25 are satisfied. Thus, the universal object of the category $\widehat{\mathbf{C}}$ exists (notation as in Corollary 3.2.25) and we denote it by $\widehat{\mathcal{Q}}$.

Before proving the next result, we note the following fact. We recall the fundamental unitary of $SU_\mu(2)$ given by $\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$ which is the matrix corresponding to the coproduct Δ on $\text{span}\{\alpha, -\mu\gamma^*\}$ as given in subsection 1.2.4. This implies that $V_{-\frac{1}{2}}^{\frac{1}{2}} = \text{span}\{\alpha, \gamma^*\}$ and $V_{\frac{1}{2}}^{\frac{1}{2}} = \text{span}\{\alpha^*, \gamma\}$.

Lemma 3.4.1. *Given a CQG \mathcal{Q} with a C^* action Φ on \mathcal{A} , the following are equivalent :*

1. (\mathcal{Q}, Φ) is an element of $\text{Obj}(\widehat{\mathbf{C}})$.
2. The action is linear, in the sense that $V_{-1/2}^{1/2}$ (equivalently, $V_{1/2}^{1/2}$) is invariant under Φ and the representation obtained by restricting Φ to $V_{1/2}^{1/2}$ is a unitary representation.
3. Φ is linear and Haar state preserving.
4. Φ keeps V_i^l invariant for each fixed l and i .

Proof : 1. \Rightarrow 2. Since Φ commutes with \widehat{D} , Φ keeps each of the eigenspaces of \widehat{D} invariant and so in particular preserves $V_{-1/2}^{1/2}$, that is Φ is linear. The condition $(h \otimes \text{id})\Phi = h(\cdot)$ implies the unitarity of the corresponding representation.

2 \Rightarrow 3. By linearity, write $\Phi(\alpha) = \alpha \otimes X + \gamma^* \otimes Y$ and $\Phi(\gamma^*) = \alpha \otimes Z + \gamma^* \otimes W$.

Firstly, Φ -invariance of $\text{Span}\{t_{i,j}^k\}$ for $k = 0$ and $k = \frac{1}{2}$ follow from the linearity and the fact that $\Phi(1) = 1$.

Next, we show that Φ keeps $\text{Span}\{t_{ij}^1 : i, j = -1, 0, 1\}$ invariant.

We recall the explicit form of the matrix $((t_{ij}^1))$ from [43]:

$$\begin{pmatrix} \alpha^{*2} & -(\mu^2 + 1)\alpha^*\gamma & -\mu\gamma^2 \\ \gamma^*\alpha^* & 1 - (\mu^2 + 1)\gamma^*\gamma & \alpha\gamma \\ -\mu\gamma^{*2} & -(\mu^2 + 1)\gamma^*\alpha & \alpha^2 \end{pmatrix}.$$

By inspection, we see that $\Phi(V_i^1) \subseteq V_i^1 \otimes \mathcal{Q}$ for $i = -1, 1$.

Hence, it is enough to check the Φ -invariance for $\alpha\gamma$ and $1 - (\mu^2 + 1)\gamma^*\gamma$.

We have

$$\begin{aligned} \Phi(\alpha\gamma) &= (\alpha \otimes X + \gamma^* \otimes Y)(\alpha^* \otimes Z^* + \gamma \otimes W^*) \\ &= \alpha\alpha^* \otimes XZ^* + \gamma^*\gamma \otimes YW^* + \alpha\gamma \otimes XW^* + \gamma^*\alpha^* \otimes YZ^* \end{aligned}$$

$$\begin{aligned}
&= \alpha\gamma \otimes XW^* + \gamma^*\alpha^* \otimes YZ^* + 1 \otimes XZ^* + (1 - (1 + \mu^2)\gamma^*\gamma) \otimes \frac{\mu^2 XZ^* - YW^*}{1 + \mu^2} \\
&\quad + 1 \otimes \frac{YW^* - \mu^2 XZ^*}{1 + \mu^2} \\
&= \alpha\gamma \otimes XW^* + \gamma^*\alpha^* \otimes YZ^* + 1 \otimes (XZ^* + \frac{YW^* - \mu^2 XZ^*}{1 + \mu^2}) + (1 - (1 + \mu^2)\gamma^*\gamma) \\
&\quad \otimes \frac{\mu^2 XZ^* - YW^*}{1 + \mu^2}.
\end{aligned}$$

Thus, comparing coefficient of 1 in $\Phi(\alpha\gamma)$, we can see that it belongs to V_0^1 if and only if $XZ^* + YW^* = 0$.

In the case of $1 - (1 + \mu^2)\gamma^*\gamma$,

$$\begin{aligned}
&\Phi(1 - (1 + \mu^2)\gamma^*\gamma) \\
&= 1 \otimes 1 - (1 + \mu^2)(\alpha\alpha^* \otimes ZZ^* + \alpha\gamma \otimes ZW^* + \gamma^*\alpha^* \otimes WZ^* + \gamma^*\gamma \otimes WW^*) \\
&= 1 \otimes 1 - (1 + \mu^2)(1 - \mu^2\gamma^*\gamma) \otimes ZZ^* - \alpha\gamma \otimes (1 + \mu^2)ZW^* - \mu\alpha^*\gamma^* \otimes (1 + \mu^2) \\
&\quad WZ^* - (1 + \mu^2)\gamma^*\gamma \otimes WW^* \\
&= 1 \otimes 1 - (1 + \mu^2).1 \otimes ZZ^* + (-1 + 1 - (1 + \mu^2)\gamma^*\gamma) \otimes -\mu^2 ZZ^* - \alpha\gamma \otimes (1 + \mu^2) \\
&\quad ZW^* - \mu\alpha^*\gamma^* \otimes (1 + \mu^2)WZ^* + (-1 + 1 - (1 + \mu^2)\gamma^*\gamma) \otimes WW^* \\
&= 1 \otimes (1 - (1 + \mu^2)ZZ^* + \mu^2 ZZ^* - WW^*) + (1 - (1 + \mu^2)\gamma^*\gamma) \otimes (-\mu^2 ZZ^* + \\
&\quad WW^*) - \alpha\gamma \otimes (1 + \mu^2)ZW^* - \mu\alpha^*\gamma^* \otimes (1 + \mu^2)WZ^*.
\end{aligned}$$

Comparing the coefficient of 1 in this case, we have the condition $1 - (1 + \mu^2)ZZ^* + \mu^2 ZZ^* - WW^* = 0$, that is, $ZZ^* + WW^* = 1$.

But these conditions follow from the unitarity of the matrix $\begin{pmatrix} X^* & Z^* \\ Y^* & W^* \end{pmatrix}$, which is nothing but the matrix corresponding to the restriction of Φ to $V_{1/2}^{1/2}$. Thus, Φ keeps $\text{Span}\{t_{ij}^1 : i, j = -1, 0, 1\}$ invariant.

Moreover, we claim that by using the recursive relations (1.2.18), (1.2.19) and the multiplication rule (1.2.23), we obtain that for all $l \geq 3/2$, $\Phi(V_i^{l+1/2}) \subseteq V_i^{l-1/2} \oplus V_i^{l+1/2}$. We prove this for $t_{i,j}^{l+\frac{1}{2}}$, $-l + \frac{1}{2} \leq i \leq l - \frac{1}{2}$, $j \leq l$ only, as the proofs of the others are exactly similar. We have

$$\begin{aligned}
&\Phi(t_{i,j}^{l+\frac{1}{2}}) \\
&= c(l, i, j)\Phi(\alpha)\Phi(t_{i+\frac{1}{2},j+\frac{1}{2}}^l) + c'(l, i, j)\Phi(\gamma)\Phi(t_{i-\frac{1}{2},j+\frac{1}{2}}^l) \\
&= c(l, i, j)\Phi(t_{-\frac{1}{2},-\frac{1}{2}}^{\frac{1}{2}})\Phi(t_{i+\frac{1}{2},j+\frac{1}{2}}^l) + c'(l, i, j)\Phi(t_{\frac{1}{2},-\frac{1}{2}}^{\frac{1}{2}})\Phi(t_{i-\frac{1}{2},j+\frac{1}{2}}^l) \\
&\in \text{Span}\{t_{-\frac{1}{2},k}^{\frac{1}{2}}.t_{i+\frac{1}{2},m}^l, t_{\frac{1}{2},k}^{\frac{1}{2}}.t_{i-\frac{1}{2},m}^l : k = \pm\frac{1}{2}, m = -l, \dots, l\} \otimes \mathcal{Q}
\end{aligned}$$

$$\subseteq V_i^{l-\frac{1}{2}} \otimes \mathcal{Q} + V_i^{l+\frac{1}{2}} \otimes \mathcal{Q}.$$

Using these observations, we conclude that Φ maps $\text{Span}\{t_{ij}^l : l \geq 1/2\}$ into itself.

So, in particular, $\text{Ker}(h) = \text{Span}\{V_i^l : i = -l, \dots, l, l \geq 1/2\}$ is invariant under Φ which (along with $\Phi(1) = 1$) implies that Φ preserves h .

3. \Rightarrow 4.

We proceed by induction. The induction hypothesis holds for $l = \frac{1}{2}$ since linearity means that $\text{span}\{\alpha, \gamma^*\}$ is invariant under Φ and hence $\text{Span}\{\alpha^*, \gamma\}$ is also invariant. The case for $l = 1$ can be checked by inspection as in the proof of $2 \Rightarrow 3$. Consider the induction hypothesis that Φ keeps V_i^k invariant for all k, i with $k \leq l$. From the proof of $2 \Rightarrow 3$ we also have for all $l \geq \frac{3}{2}$, $\Phi(V_i^{l+1/2}) \subseteq V_i^{l-1/2} \oplus V_i^{l+1/2}$, by using linearity only. Thus, $\tilde{\Phi}$ leaves invariant the Hilbert \mathcal{Q} module $(V_i^{l-\frac{1}{2}} \oplus V_i^{l+\frac{1}{2}}) \otimes \mathcal{Q}$, and is a unitary there since Φ is Haar-state preserving. Since $\tilde{\Phi}$ leaves invariant $V_i^{l-\frac{1}{2}} \otimes \mathcal{Q}$ by the induction hypothesis, it must keep its orthocomplement, $V_i^{l+\frac{1}{2}} \otimes \mathcal{Q}$ invariant as well.

4. \Rightarrow 3.

The fact that Φ keeps V_i^l invariant for $l = 1/2$ will imply that Φ is linear. The proof of Haar state preservation is exactly the same as in $2 \Rightarrow 3$.

4 \Rightarrow 1.

That Φ preserves the Haar state follows from arguments used in the proof of the implication $2 \Rightarrow 3$. Since $\mathcal{A}_{00} = \text{Span}\{t_{ij}^l : l \geq 0, i, j = -l, \dots, l\}$, and Φ keeps each V_i^l invariant, it is obvious that $\Phi(\mathcal{A}_{00}) \subseteq \mathcal{A}_{00} \otimes_{\text{alg}} \mathcal{Q}_0$ and $\Phi\hat{D} = (\hat{D} \otimes \text{id})\Phi$.

□

By Lemma 3.4.1, we have identified the category $\hat{\mathbf{C}}$ with the category of CQG having C^* actions on $SU_\mu(2)$ satisfying condition 3. of Lemma 3.4.1. Let the universal object of this category be denoted by $(\hat{\mathcal{Q}}, \Gamma)$.

Then by linearity we can write:

$$\Gamma(\alpha) = \alpha \otimes A + \gamma^* \otimes B,$$

$$\Gamma(\gamma^*) = \alpha \otimes C + \gamma^* \otimes D.$$

Now we shall exploit the fact that Γ is a $*$ -homomorphism to get relations satisfied by A, B, C, D where $\hat{\mathcal{Q}}$ is generated as a C^* algebra by the elements A, B, C, D .

Lemma 3.4.2.

$$A^*A + CC^* = 1, \tag{3.4.1}$$

$$A^*A + \mu^2 CC^* = B^*B + DD^*, \quad (3.4.2)$$

$$A^*B = -\mu DC^*, \quad (3.4.3)$$

$$B^*A = -\mu CD^*. \quad (3.4.4)$$

Proof : The proof follows from the relation (1.2.10) by comparing coefficients of $1, \gamma^*\gamma, \alpha^*\gamma^*$ and $\alpha\gamma$ respectively. \square

Lemma 3.4.3.

$$AA^* + \mu^2 CC^* = 1, \quad (3.4.5)$$

$$BB^* + \mu^2 DD^* = \mu^2.1, \quad (3.4.6)$$

$$BA^* = -\mu^2 DC^*. \quad (3.4.7)$$

Proof : From the equation (1.2.11) by equating coefficients of 1 and $\alpha^*\gamma^*$, we get respectively (3.4.5) and (3.4.7) whereas (3.4.6) is obtained by equating coefficients of $\gamma^*\gamma$ and using (3.4.5). \square

Lemma 3.4.4.

$$C^*C = CC^*, \quad (3.4.8)$$

$$(1 - \mu^2)C^*C = D^*D - DD^*, \quad (3.4.9)$$

$$C^*D = \mu DC^*. \quad (3.4.10)$$

Proof : The proof follows from the equation (1.2.12) by comparing the coefficients of $1, \gamma^*\gamma, \alpha^*\gamma^*$, respectively. \square

Lemma 3.4.5.

$$-\mu^2 AC^* + BD^* - \mu D^*B + \mu C^*A = 0, \quad (3.4.11)$$

$$AC^* = \mu C^*A, \quad (3.4.12)$$

$$BC^* = C^*B, \quad (3.4.13)$$

$$AD^* = D^*A. \quad (3.4.14)$$

Proof : The proof follows from the equation (1.2.13) comparing the coefficients of $\gamma^*\gamma, 1, \alpha^*\gamma^*$ and $\alpha\gamma$ respectively. \square

Lemma 3.4.6.

$$AC = \mu CA, \quad (3.4.15)$$

$$BD = \mu DB, \quad (3.4.16)$$

$$AD - \mu CB = DA - \mu^{-1}BC. \quad (3.4.17)$$

Proof : The proof follows from (1.2.14) from the coefficients of $\alpha^2, \gamma^{*2}, \gamma^* \alpha$ respectively. \square

Now we consider the antipode, say κ .

From the condition $(h \otimes \text{id})\Gamma(a) = h(a).1$, we see that Γ induces a unitary representation of the compact quantum group via $\tilde{\Gamma}(a \otimes q) = \Gamma(a)(1 \otimes q)$.

Now, the restriction of this unitary representation to the orthonormal set

$$\left\{ \sqrt{\frac{1+\mu^2}{\mu^2}}\alpha, \sqrt{1+\mu^2}\gamma^* \right\} \text{ is given by the matrix : } \begin{pmatrix} A & \mu C \\ \mu^{-1}B & D \end{pmatrix}.$$

Similarly, with respect to the orthonormal set $\{\sqrt{1+\mu^2}\alpha^*, \sqrt{1+\mu^2}\gamma\}$, this representation is given by the matrix: $\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}$.

Thus, we have:

$$\kappa(A) = A^*, \kappa(D) = D^*, \kappa(C) = \mu^{-2}B^*, \kappa(B) = \mu^2C^*, \kappa(A^*) = A, \kappa(C^*) = B, \kappa(B^*) = C, \kappa(D^*) = D.$$

Lemma 3.4.7.

$$AB = \mu BA, \quad (3.4.18)$$

$$CD = \mu DC, \quad (3.4.19)$$

$$BC^* = C^*B. \quad (3.4.20)$$

Proof : The relations (3.4.18), (3.4.19), (3.4.20) follow by applying κ to the equations (3.4.15), (3.4.16) and (3.4.13) respectively.

Lemma 3.4.8. *There exists a $*$ -homomorphism $\phi : U_\mu(2) \rightarrow \hat{\mathcal{Q}}$ defined by $\phi(u_{11}) = A$, $\phi(u_{12}) = \mu C$, $\phi(u_{21}) = \mu^{-1}B$, $\phi(u_{22}) = D$.*

Proof : It is enough to check that the defining relations of $U_\mu(2)$ are satisfied.

1. $\phi(u_{11}u_{12}) = \phi(\mu u_{12}u_{11}) \Leftrightarrow \phi(u_{11})\phi(u_{12}) = \mu\phi(u_{12})\phi(u_{11}) \Leftrightarrow A(\mu C) = \mu(\mu C)A \Leftrightarrow AC = \mu CA$ which is the same as (3.4.15).

2. $\phi(u_{11}u_{21}) = \phi(\mu u_{21}u_{11}) \Leftrightarrow A(\mu^{-1}B) = \mu(\mu^{-1}B)A \Leftrightarrow AB = \mu BA$ which is the same as equation (3.4.18).

3. $\phi(u_{12}u_{22}) = \phi(\mu u_{22}u_{12}) \Leftrightarrow \mu CD = \mu D(\mu C) \Leftrightarrow CD = \mu DC$ which is the same as equation (3.4.19).

4. $\phi(u_{21}u_{22}) = \phi(\mu u_{22}u_{21}) \Leftrightarrow \mu^{-1}BD = \mu D\mu^{-1}B \Leftrightarrow BD = \mu DB$ which is the same as equation (3.4.16).

5. $\phi(u_{12}u_{21}) = \phi(u_{21}u_{12}) \Leftrightarrow \mu C\mu^{-1}B = \mu^{-1}B\mu C \Leftrightarrow CB = BC$.

Now, $BC^* = C^*B$ follows from equation (3.4.20). But by (3.4.8), C is normal, which implies $BC = CB$.

$$6. \phi(u_{11}u_{22} - u_{22}u_{11}) = (\mu - \mu^{-1})\phi(u_{12}u_{21}) \Leftrightarrow AD - DA = (\mu - \mu^{-1})\mu C\mu^{-1}B.$$

From (3.4.17), we have $AD - DA = \mu CB - \mu^{-1}BC = (\mu - \mu^{-1})CB$, using $BC = CB$.

□

Lemma 3.4.9. *The equations (3.4.1) - (3.4.17) are true when A, B, C, D are replaced by $u_{11}, \mu u_{21}, \mu^{-1}u_{12}$ and u_{22} respectively.*

Proof : We check some of the relations (3.4.1) - (3.4.17 by using the facts that D_μ is a central element of $U_\mu(2)$, $\kappa(u_{ij}) = u_{ji}^*$ ((1.2.8)), Proposition 1.2.26, the equations (1.2.2) - (1.2.7) and (1.2.9). The proofs of the others are exactly similar.

Proof for (3.4.1) that is, $u_{11}^*u_{11} + (\mu^{-1}u_{12})(\mu^{-1}u_{12})^* = 1$.

$$\begin{aligned} u_{11}^*u_{11} + \mu^{-2}u_{12}u_{12}^* &= u_{22}D_\mu^{-1}u_{11} + \mu^{-2}u_{12}(-\mu u_{21}D_\mu^{-1}) \\ &= (u_{22}u_{11} - \mu^{-1}u_{12}u_{21})D_\mu^{-1} = D_\mu D_\mu^{-1} = 1. \end{aligned}$$

Proof for (3.4.2) that is, $u_{11}^*u_{11} + \mu^2(\mu^{-1}u_{12})(\mu^{-1}u_{12})^* - ((\mu u_{21})^*\mu u_{21} + u_{22}u_{22}^*) = 0$.

$$\begin{aligned} &u_{11}^*u_{11} + \mu^2(\mu^{-1}u_{12})(\mu^{-1}u_{12})^* - ((\mu u_{21})^*\mu u_{21} + u_{22}u_{22}^*) \\ &= \kappa(u_{11})u_{11} + u_{12}\kappa(u_{21}) - (\mu^2\kappa(u_{12})u_{21} + u_{22}\kappa(u_{22})) \\ &= (u_{22}u_{11} - \mu u_{12}u_{21})D_\mu^{-1} - (-\mu u_{12}u_{21} + u_{22}u_{11})D_\mu^{-1} \\ &= 0. \end{aligned}$$

Proof for (3.4.6) that is, $\mu^2u_{21}u_{21}^* + \mu^2u_{22}u_{22}^* - \mu^2.1 = 0$.

$$\begin{aligned} &\mu^2u_{21}u_{21}^* + \mu^2u_{22}u_{22}^* - \mu^2.1 \\ &= \mu^2(u_{21}\kappa(u_{12}) + u_{22}\kappa(u_{22}) - 1) \\ &= \mu^2(u_{21}(-\mu^{-1}u_{12}D_\mu^{-1}) + u_{22}u_{11}D_\mu^{-1} - 1) \\ &= \mu^2((u_{22}u_{11} - \mu^{-1}u_{21}u_{12})D_\mu^{-1} - 1) \\ &= \mu^2(D_\mu D_\mu^{-1} - 1) \\ &= 0. \end{aligned}$$

Proof for (3.4.7) that is, $u_{21}u_{11}^* + u_{22}u_{12}^* = 0$.

$$\begin{aligned}
& u_{21}u_{11}^* + u_{22}u_{12}^* \\
&= u_{21}\kappa(u_{11}) + u_{22}\kappa(u_{21}) \\
&= u_{21}u_{22}D_\mu^{-1} - \mu u_{22}u_{21}D_\mu^{-1} \\
&= (u_{21}u_{22} - \mu u_{22}u_{21})D_\mu^{-1} \\
&= 0.
\end{aligned}$$

Proof for (3.4.9) that is, $(1 - \mu^2)u_{12}^*u_{12} - \mu^2(u_{22}^*u_{22} - u_{22}u_{22}^*) = 0$.

$$\begin{aligned}
& (1 - \mu^2)u_{12}^*u_{12} - \mu^2(u_{22}^*u_{22} - u_{22}u_{22}^*) \\
&= (1 - \mu^2)\kappa(u_{21})u_{12} - \mu^2(\kappa(u_{22})u_{22} - u_{22}\kappa(u_{22})) \\
&= -\mu(1 - \mu^2)(u_{21}u_{12}D_\mu^{-1}) - \mu^2(u_{11}u_{22}D_\mu^{-1} - u_{22}u_{11}D_\mu^{-1}) \\
&= -\mu[(1 - \mu^2)(u_{21}u_{12}D_\mu^{-1}) - \mu(u_{22}u_{11}D_\mu^{-1} - u_{11}u_{22}D_\mu^{-1})] \\
&= -\mu[(1 - \mu^2)u_{21}u_{12}D_\mu^{-1} - \mu(\mu^{-1} - \mu)u_{12}u_{21}D_\mu^{-1}] \\
&= -\mu(1 - \mu^2)(u_{12}u_{21} - u_{12}u_{21})D_\mu^{-1} \\
&= 0.
\end{aligned}$$

Proof for (3.4.10) that is, $u_{12}^*u_{22} - \mu u_{22}u_{12}^* = 0$.

$$\begin{aligned}
& u_{12}^*u_{22} - \mu u_{22}u_{12}^* \\
&= \kappa(u_{21})u_{22} - \mu u_{22}\kappa(u_{21}) \\
&= \mu^2 u_{22}u_{21}D_\mu^{-1} - \mu u_{21}u_{22}D_\mu^{-1} \\
&= \mu(\mu u_{22}u_{21} - u_{21}u_{22})D_\mu^{-1} \\
&= 0.
\end{aligned}$$

Proof for (3.4.11) that is, $-\mu u_{11}u_{12}^* + \mu u_{21}u_{22}^* - \mu^2 u_{22}^*u_{21} + u_{12}^*u_{11} = 0$.

$$\begin{aligned}
& -\mu u_{11}u_{12}^* + \mu u_{21}u_{22}^* - \mu^2 u_{22}^*u_{21} + u_{12}^*u_{11} \\
&= -\mu u_{11}\kappa(u_{21}) + \mu u_{21}\kappa(u_{22}) - \mu^2 \kappa(u_{22})u_{21} + \kappa(u_{21})u_{11} \\
&= -\mu u_{11}(-\mu u_{21}D_\mu^{-1}) + \mu u_{21}(u_{11}D_\mu^{-1}) - \mu^2 u_{11}u_{21}D_\mu^{-1} - \mu u_{21}u_{11}D_\mu^{-1} \\
&= \mu^2(u_{11}u_{21} - u_{11}u_{21})D_\mu^{-1} + \mu(u_{21}u_{11} - u_{21}u_{11})D_\mu^{-1} \\
&= 0.
\end{aligned}$$

Proof for (3.4.17) that is, $u_{11}u_{22} - \mu(\mu^{-1}u_{12})\mu u_{21} = u_{22}u_{11} - \mu^{-1}(\mu u_{21})(\mu^{-1}u_{12})$.

$$\begin{aligned} & u_{11}u_{22} - \mu(\mu^{-1}u_{12})\mu u_{21} - u_{22}u_{11} + \mu^{-1}(\mu u_{21})(\mu^{-1}u_{12}) \\ &= u_{11}u_{22} - \mu u_{12}u_{21} - u_{22}u_{11} + \mu^{-1}u_{21}u_{12} \\ &= 0. \end{aligned}$$

□

Lemma 3.4.10. *There is a C^* action Ψ of $U_\mu(2)$ on $SU_\mu(2)$ such that $(U_\mu(2), \Psi)$ is an object of $\text{Obj}(\widehat{\mathbf{C}})$ and Ψ is given by :*

$$\Psi(\alpha) = \alpha \otimes u_{11} + \gamma^* \otimes \mu u_{21},$$

$$\Psi(\gamma^*) = \alpha \otimes \mu^{-1}u_{12} + \gamma^* \otimes u_{22}.$$

Proof : The homomorphism conditions are exactly the conditions (3.4.1) - (3.4.17) with A, B, C, D replaced by $u_{11}, \mu u_{21}, \mu^{-1}u_{12}$ and u_{22} respectively which are true by Lemma 3.4.9.

Clearly, Ψ keeps $V_{-1/2}^{1/2}$ invariant and the corresponding representation is a unitary.

It follows from Lemma 3.4.1 that $(U_\mu(2), \Psi)$ is an object of $\widehat{\mathbf{C}}$.

□

Corollary 3.4.11. *There exists a surjective CQG morphism from $\widehat{\mathcal{Q}}$ to $U_\mu(2)$ sending $A, \mu C, \mu^{-1}B$, and D to u_{11}, u_{12}, u_{21} and u_{22} respectively.*

Theorem 3.4.12. *We have $\widehat{\mathcal{Q}} \cong U_\mu(2)$ and hence $\widetilde{QISO^+}(D) \cong U_\mu(2) * C(\mathbb{T})$.*

Proof : The first part follows from Lemma 3.4.8 and Corollary 3.4.11 and the second part follows from Theorem 3.2.22. □

3.4.2 A commutative example : spectral triple on \mathbb{T}^2

We consider the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ on \mathbb{T}^2 given by $\mathcal{A}^\infty = C^\infty(\mathbb{T}^2)$, $\mathcal{H} = L^2(\mathbb{T}^2) \oplus L^2(\mathbb{T}^2)$ and $D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}$,

where we view $C(\mathbb{T}^2)$ as the universal C^* algebra generated by two commuting unitaries U and V , and d_1 and d_2 are derivations on \mathcal{A}^∞ defined by :

$$d_1(U) = U, \quad d_1(V) = 0, \quad d_2(U) = 0, \quad d_2(V) = V. \quad (3.4.21)$$

The vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ form an orthonormal basis of the eigenspace corresponding to the eigenvalue zero.

The Laplacian in the sense of chapter 2 exists in this case, and is given by $\mathcal{L}(U^m V^n) = -(m^2 + n^2)U^m V^n$. We recall that we denote the quantum isometry group from the Laplacian \mathcal{L} in the sense of [30] by $QISO^{\mathcal{L}^D}$.

Lemma 3.4.13. *Let $(\tilde{\mathcal{Q}}, W)$ be an object of $\mathbf{Q}'(D)$. Then the $*$ -homomorphism $\alpha = \alpha_W$ must be of the following form:*

$$\alpha(U) = U \otimes z_1, \quad (3.4.22)$$

$$\alpha(V) = V \otimes z_2, \quad (3.4.23)$$

where z_1, z_2 are two commuting unitaries.

Proof: We denote the the maximal Woronowicz C^* subalgebra of $\tilde{\mathcal{Q}}$ which acts on $C(\mathbb{T}^2)$ faithfully by \mathcal{Q} .

We observe that $D^2(ae_i) = \mathcal{L}(a)e_i$ for $i = 1, 2$. Now, W commutes with D implies that W commutes with D^2 as well. Using this, we can show that $(\mathcal{L} \otimes \text{id})\alpha(a)e_i = \alpha\mathcal{L}(a)e_i$, $i = 1, 2$. As the pair $\{e_1, e_2\}$ is together separating for $C(\mathbb{T}^2)$, we conclude that α commutes with the Laplacian \mathcal{L} . Therefore, \mathcal{Q} is a quantum subgroup of $QISO^{\mathcal{L}^D}$. From Theorem 2.2.17, we conclude that $QISO^{\mathcal{L}^D} = C(\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes \mathbf{Z}_2))$. Thus \mathcal{Q} must be of the form $C(G)$ for a classical subgroup G of the orientation preserving isometry group of \mathbb{T}^2 , which is \mathbb{T}^2 itself and whose (co)action is given by $U \mapsto U \otimes U$ and $V \mapsto V \otimes V$. \square

Theorem 3.4.14. *The universal $CQG \widetilde{QISO}^+(C^\infty(\mathbb{T}^2), \mathcal{H}, D)$ exists and is isomorphic with $C(\mathbb{T}^2) * C(\mathbb{T}) \cong C^*(\mathbf{Z}^2 * \mathbf{Z})$ (as a CQG). Moreover, $QISO^+$ of this spectral triple is isomorphic with $C(\mathbb{T}^2)$.*

Proof: Let $(\tilde{\mathcal{Q}}, W)$ be an object in $\mathbf{Q}'(D)$ as in Lemma 3.4.13. Since $\{e_1, e_2\}$ is an orthonormal basis for an eigenspace of D , we must have

$$W(e_1) = e_1 \otimes q_{11} + e_2 \otimes q_{12}, \quad (3.4.24)$$

$$W(e_2) = e_1 \otimes q_{21} + e_2 \otimes q_{22}, \quad (3.4.25)$$

for some q_{ij} in $\tilde{\mathcal{Q}}$.

We now make use of the equation $(D \otimes \text{id})\widetilde{W}(Ue_1 \otimes 1) = \widetilde{W}(D \otimes \text{id})(Ue_1 \otimes 1)$. Let

z_1, z_2 are as in Lemma 3.4.13. We compute

$$\begin{aligned}
& (D \otimes \text{id})\widetilde{W}(Ue_1 \otimes 1) \\
&= (D \otimes \text{id})(\alpha(U)\widetilde{W}(e_1 \otimes 1)) \\
&= (D \otimes \text{id})(U \otimes z_1)(e_1 \otimes q_{11} + e_2 \otimes q_{12}) \\
&= (D \otimes \text{id})(Ue_1 \otimes z_1q_{11} + Ue_2 \otimes z_1q_{12}) \\
&= Ue_2 \otimes z_1q_{11} + Ue_1 \otimes z_1q_{12}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \widetilde{W}(D \otimes \text{id})(Ue_1 \otimes 1) \\
&= \widetilde{W}(Ue_2 \otimes 1) \\
&= \widetilde{W}(U \otimes \text{id})\widetilde{W}^*\widetilde{W}(e_2 \otimes 1) \\
&= \alpha(U)\widetilde{W}(e_2 \otimes 1) \\
&= (U \otimes z_1)(e_1 \otimes q_{21} + e_2 \otimes q_{22}) \\
&= Ue_1 \otimes z_1q_{21} + Ue_2 \otimes z_1q_{22}.
\end{aligned}$$

By comparing coefficients of Ue_1 and Ue_2 in the both sides of the equality $(D \otimes \text{id})W(Ue_1) = WDUe_1$, we have,

$$z_1q_{12} = z_1q_{21} \tag{3.4.26}$$

and

$$z_1q_{11} = z_1q_{22}. \tag{3.4.27}$$

Since z_1 is a unitary, we have $q_{11} = q_{22}$ and $q_{12} = q_{21}$.

Similarly, from the relation $(D \otimes I)W(Ve_1) = WDVe_1$, we have $q_{12} = -q_{21}, q_{22} = q_{11}$.

By the above two sets of relations, we obtain :

$$q_{12} = q_{21} = 0, \quad q_{11} = q_{22} = q \text{ (say)}.$$

But the matrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ is a unitary in $M_2(\widetilde{\mathcal{Q}})$, so q is a unitary.

Moreover, we note that $W(ae_i) = \alpha(a)W(e_i)$ for all a in $C^\infty(\mathbb{T}^2)$. Using Lemma 3.4.13 and the above observations, we deduce that any CQG which has a unitary representation commuting with the Dirac operator is a quantum subgroup of $C(\mathbb{T}^2) * C(\mathbb{T})$.

On the other hand, $C(\mathbb{T}^2) * C(\mathbb{T})$ has a unitary representation commuting with D , given by the formulae (3.4.22) - (3.4.25) taking $q_{12} = q_{21} = 0, \quad q_{11} = q_{22} = q'$ where q' is the generator of $C(\mathbb{T})$ and z_1, z_2 to be the generator of $C(\mathbb{T}^2)$. This completes the proof. \square

Remark 3.4.15. The canonical grading on $C(\mathbb{T}^2)$ is given by the operator $(\text{id} \otimes \gamma)$ on $L^2(\mathbb{T}^2 \otimes \mathbb{C}^2)$ where γ is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The representation of $C(\mathbb{T}^2) * C(\mathbb{T})$ clearly commutes with the grading operator and hence is isomorphic with $\widetilde{QISO}(C(\mathbb{T}^2), L^2(\mathbb{T}^2 \otimes \mathbb{C}^2), D, \gamma)$.

Remark 3.4.16. This example shows that the conditions of Theorem 3.2.22 are not necessary for the existence of $QISO^+$.

3.5 $QISO^+$ for zero dimensional manifolds

3.5.1 Inductive limit construction for quantum isometry groups

In this section we use the limiting construction for an inductive system of compact quantum groups (Lemma 1.2.25) and give an application for quantum isometry groups which is fundamental for the results of the next section.

The next theorem connects the inductive construction done in Lemma 1.2.25 with some specific quantum isometry groups.

Theorem 3.5.1. Suppose that \mathcal{A} is a C^* -algebra acting on a Hilbert space \mathcal{H} and that D is a (densely defined) self adjoint operator on \mathcal{H} with compact resolvent, such that D has a one-dimensional eigenspace spanned by a vector ξ which is cyclic and separating for \mathcal{A} . Let $(\mathcal{A}_n^\infty)_{n \in \mathbb{N}}$ be an increasing net of a unital $*$ -subalgebras of \mathcal{A} and put $\mathcal{A}^\infty = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n^\infty$. Suppose that \mathcal{A}^∞ is dense in \mathcal{A} and that for each $a \in \mathcal{A}^\infty$ the commutator $[D, a]$ is densely defined and bounded. Additionally put $\mathcal{H}_n = \overline{\mathcal{A}_n^\infty \xi}$, let P_n denote the orthogonal projection on \mathcal{H}_n and assume that each P_n commutes with D . Then each $(\mathcal{A}_n^\infty, \mathcal{H}_n, D|_{\mathcal{H}_n})$ is a spectral triple satisfying the conditions of Theorem 3.2.13, there exist natural compatible CQG morphisms $\pi_{m,n} : \widetilde{QISO}^+(\mathcal{A}_m^\infty, \mathcal{H}_m, D|_{\mathcal{H}_m}) \rightarrow \widetilde{QISO}^+(\mathcal{A}_n^\infty, \mathcal{H}_n, D|_{\mathcal{H}_n})$ ($n, m \in \mathbb{N}, m \leq n$) and

$$\widetilde{QISO}^+(\mathcal{A}^\infty, \mathcal{H}, D) = \lim_{n \in \mathbb{N}} \widetilde{QISO}^+(\mathcal{A}_n^\infty, \mathcal{H}_n, D|_{\mathcal{H}_n}).$$

Similar conclusions hold if we replace everywhere above \widetilde{QISO}^+ by $QISO^+$.

Proof: We prove the assertion corresponding to \widetilde{QISO}^+ only, since the proof for $QISO^+$ follows by very similar arguments. Let us denote $\widetilde{QISO}^+(\mathcal{A}_n^\infty, \mathcal{H}_n, D_n)$ by \mathcal{S}_n and the corresponding unitary representation (in \mathcal{H}_n) by U_n . Let us denote the category of compact quantum groups acting by orientation preserving isometries on $(\mathcal{A}_n^\infty, \mathcal{H}_n, D|_{\mathcal{H}_n})$ and $(\mathcal{A}^\infty, \mathcal{H}, D)$ respectively by \mathbf{Q}_n and \mathbf{Q} .

Since U_n is a unitary which commutes with $D_n \equiv D|_{\mathcal{H}_n}$ and hence preserves the eigenspaces of D_n , it restricts to a unitary representation of \mathcal{S}_n on each \mathcal{H}_m for $m \leq n$. In other words, $(\mathcal{S}_n, U_n|_{\mathcal{H}_m}) \in \text{Obj}(\mathbf{Q}_m)$, and by the universality of \mathcal{S}_m there exists a compact quantum group morphism, say, $\pi_{m,n} : \mathcal{S}_m \rightarrow \mathcal{S}_n$ such that $(\text{id} \otimes \pi_{m,n})U_m|_{\mathcal{H}_m} = U_n|_{\mathcal{H}_m}$.

Let $p \leq m \leq n$. Then we have $(\text{id} \otimes \pi_{m,n}\pi_{p,m})U_p|_{\mathcal{H}_p} = U_n|_{\mathcal{H}_p}$. It follows by the uniqueness of the map $\pi_{p,n}$ that $\pi_{p,n} = \pi_{m,n}\pi_{p,m}$, that is $(\mathcal{S}_n)_{n \in \mathbb{N}}$ forms an inductive system of compact quantum groups satisfying the assumptions of Lemma 1.2.25. Denote by \mathcal{S}_∞ the inductive limit CQG obtained in that lemma, with $\pi_{n,\infty} : \mathcal{S}_n \rightarrow \mathcal{S}$ denoting the corresponding CQG morphisms. The family of formulas $U|_{\mathcal{H}_n} := (\text{id} \otimes \pi_{n,\infty}) \circ U_n$ combine to define a unitary representation U of \mathcal{S}_∞ on \mathcal{H} . It is also easy to see from the construction that U commutes with D . This means that $(\mathcal{S}_\infty, U) \in \text{Obj}(\mathbf{Q})$, hence there exists a unique surjective CQG morphism from $\mathcal{S} := \widetilde{QISO^+(\mathcal{A}^\infty, \mathcal{H}, D)}$ to \mathcal{S}_∞ identifying \mathcal{S}_∞ as a quantum subgroup of \mathcal{S} .

The proof will now be complete if we can show that there is a surjective CQG morphism in the reverse direction, identifying \mathcal{S} as a quantum subgroup of \mathcal{S}_∞ . This can be deduced from Lemma 1.2.25 by using the universality property of the inductive limit. Indeed, for each $n \in \mathbb{N}$ the unitary representation, say V_n , of $\widetilde{QISO^+(\mathcal{A}^\infty, \mathcal{H}, D)}$ restricts to \mathcal{H}_n and commutes with D on that subspace, thus inducing a CQG morphism ρ_n from $\mathcal{S}_n = \widetilde{QISO^+(\mathcal{A}_n^\infty, \mathcal{H}_n, D_n)}$ into \mathcal{S} . The family of morphisms $(\rho_n)_{n \in \mathbb{N}}$ satisfies the compatibility conditions required in Lemma 1.2.25. It remains to show that the induced CQG morphism ρ_∞ from \mathcal{S}_∞ into \mathcal{S} is surjective. By the faithfulness of the representation V of $\widetilde{QISO^+(\mathcal{A}^\infty, \mathcal{H}, D)}$, we know that the span of matrix elements corresponding to all V_n forms a norm-dense subset of \mathcal{S} . As the range of ρ_n contains the matrix elements corresponding to $V_n = V|_{\mathcal{H}_n}$, the proof of surjectivity of ρ_∞ is finished. \square

The assumptions of the theorem might seem very restrictive. In the next section however we will describe a natural family of spectral triples on AF -algebras, constructed in [16], for which we have exactly the situation as above.

3.5.2 Quantum isometry groups for spectral triples on AF algebras

We first recall the construction of natural spectral triples on AF algebras due to E. Christensen and C. Ivan ([16]). Let \mathcal{A} be a unital AF C^* -algebra, the norm closure of an increasing sequence $(\mathcal{A}_n)_{n \in \mathbb{N}}$ of finite dimensional C^* -algebras. We always put $\mathcal{A}_0 = \mathbb{C}1_{\mathcal{A}}$, $\mathcal{A}^\infty = \bigcup_{n=1}^\infty \mathcal{A}_n$ and assume that the unit in each \mathcal{A}_n is the unit of \mathcal{A} . Suppose that \mathcal{A} is acting on a Hilbert space \mathcal{H} and that $\xi \in \mathcal{H}$ is a separating and cyclic unit vector for \mathcal{A} . Let P_n denote the orthogonal projection onto the subspace

$\mathcal{H}_n := \mathcal{A}_n \xi$ of \mathcal{H} and write $Q_0 = P_0 = P_{\mathbb{C}\xi}$, $Q_n = P_n - P_{n-1}$ for $n \in \mathbb{N}$. There exists a (strictly increasing) sequence of real numbers $(\alpha_n)_{n=1}^\infty$ such that the self adjoint operator $D = \sum_{n \in \mathbb{N}} \alpha_n Q_n$ yields a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$. Due to the existence of a cyclic and separating vector the quantum group of orientation preserving isometries exists by Theorem 3.2.22.

In [16], the following fact was also observed:

Proposition 3.5.2. *If \mathcal{A} is infinite-dimensional and $p > 0$ then one can choose $(\alpha_n)_{n=1}^\infty$ in such a way that the spectral triple is p -summable. For this reasons, the spectral triple should be thought of as **0-dimensional noncommutative manifolds**.*

Note that for each $n \in \mathbb{N}$ by restricting we obtain a (finite-dimensional) spectral triple $(\mathcal{A}_n, \mathcal{H}_n, D|_{\mathcal{H}_n})$. As we are precisely in the framework of Theorem 3.5.1, to compute $QISO^+(\mathcal{A}^\infty, \mathcal{H}, D)$ we need to understand the quantum isometry groups $QISO^+(\mathcal{A}_n^\infty, \mathcal{H}_n, D|_{\mathcal{H}_n})$ and embeddings relating them. To simplify the notation we will write $\mathcal{S}_n := QISO^+(\mathcal{A}_n, \mathcal{H}_n, D|_{\mathcal{H}_n})$.

We begin with some general observations.

Lemma 3.5.3. *Let $\mathcal{QU}_{\mathcal{A}_n, \omega_\xi}$ denote the universal quantum group acting on \mathcal{A}_n and preserving the (faithful) state on \mathcal{A}_n given by vector ξ (see [60]). There exists a CQG morphism from $\mathcal{QU}_{\mathcal{A}_n, \omega_\xi}$ to \mathcal{S}_n .*

Proof: The proof is based on considering the spectral triple given by $(\mathcal{A}_n, \mathcal{H}_n, D'_n)$, where $D'_n = P_n - P_0$. It is then easy to see that $QISO^+(\mathcal{A}_n, \mathcal{H}_n, D'_n)$ is isomorphic to the universal compact quantum group acting on \mathcal{A}_n and preserving ω_ξ . On the other hand universality assures the existence of the CQG morphism from $QISO^+(\mathcal{A}_n, \mathcal{H}_n, D'_n)$ to \mathcal{S}_n . \square

Lemma 3.5.4. *Assume that each \mathcal{A}_n is commutative, $\mathcal{A}_n = \mathbb{C}^{k_n}$, $n \in \mathbb{N}$. There exists a CQG morphism from \mathcal{QU}_{k_n} to \mathcal{S}_{k_n} , where \mathcal{QU}_{k_n} denotes the universal quantum group acting on k_n points ([60]).*

Proof: We observe that for any measure μ on the set $\{1, \dots, k_n\}$ which has full support there is a natural CQG morphism from \mathcal{QU}_{k_n} to $\mathcal{QU}_{\mathbb{C}^{k_n}, \mu}$. In case when μ is uniformly distributed, we simply have $\mathcal{QU}_{\mathbb{C}^{k_n}, \mu} = \mathcal{QU}_{k_n}$, as follows from Lemma 1.2.33. \square

Let $\alpha_n : \mathcal{A}_n \rightarrow \mathcal{A}_n \otimes \mathcal{S}_n$ denote the universal action (on the n -th level). Then we have the following important property, being the direct consequence of the Theorem 3.5.1. We have

$$\alpha_{n+1}(\mathcal{A}_n) \subset \mathcal{A}_n \otimes \mathcal{S}_{n+1} \quad (3.5.1)$$

(where we identified \mathcal{A}_n with a subalgebra of \mathcal{A}_{n+1}) and \mathcal{S}_n is generated exactly by these coefficients of \mathcal{S}_{n+1} which appear in the image of \mathcal{A}_n under α_{n+1} . This in conjunction with the previous lemma suggests the strategy for computing relevant quantum isometry groups inductively. Suppose that we have determined the generators of \mathcal{S}_n . Then \mathcal{S}_{n+1} is generated by generators of \mathcal{S}_n and these of the $\mathcal{QU}_{\mathcal{A}_n, \omega_\xi}$, with the only additional relations provided by the equation (3.5.1).

This will be used below to determine the concrete form of relations determining \mathcal{S}_n for the commutative AF algebras.

Before stating the next result, we fix some notations. Let \mathcal{A}_n be a sequence of commutative finite dimensional C^* algebras as above. Let $\mathcal{A}_n = C(X_n)$ where $X_n = \{x_1, x_2, \dots, x_m\}$. Dualizing the embedding from \mathcal{A}_n to \mathcal{A}_{n+1} , there is a surjective map, say $f_{n+1,n}$ from X_{n+1} to X_n . Let l_i denote the cardinality of the set $\{x \in X_{n+1} : f_{n+1,n}(x) = x_i\}$. Thus the embedding of \mathcal{A}_n into \mathcal{A}_{n+1} is determined by the sequence $\{l_i : i = 1, 2, \dots, m\}$. We note that the cardinality of X_{n+1} equals $\sum_{i=1}^m l_i$. Moreover, a basis of \mathcal{A}_{n+1} is given by $\{e_{i,r_i} : r_i \in \{1, 2, \dots, l_i\}\}$ where e_{i,r_i} is the indicator function of an element y of X_{n+1} such that $f_{n+1,n}(y) = x_i$ and y is the r_i th element in X_{n+1} belonging to $f_{n+1,n}^{-1}\{x_i\}$.

Lemma 3.5.5. *Let \mathcal{A} be a commutative AF algebra. Suppose that \mathcal{A}_n is isomorphic to \mathbb{C}^m and the embedding of \mathcal{A}_n into \mathcal{A}_{n+1} is given by a sequence $(l_i)_{i=1}^m$. Let $m' = \sum_{i=1}^m l_i$. Suppose that the ‘copy’ of \mathcal{QU}_m in \mathcal{S}_n is given by the family of projections $a_{i,j}$ ($i, j \in \{1, \dots, m\}$) and that the ‘copy’ of $\mathcal{QU}_{m'}$ in \mathcal{S}_{n+1} is given by the family of projections $a_{(i,r_i),(j,s_j)}$ ($i, j \in \{1, \dots, m\}$, $r_i \in \{1, \dots, l_i\}$, $s_j \in \{1, \dots, l_j\}$). Then the formula (3.5.1) is equivalent to the following system of equalities:*

$$a_{i,j} = \sum_{r_i=1}^{l_i} a_{(i,r_i),(j,s_j)} \quad (3.5.2)$$

for each $i, j \in \{1, \dots, m\}, s_j \in \{1, \dots, l_j\}$.

Proof: We have (for the universal action $\alpha : \mathcal{A}_n \rightarrow \mathcal{A}_n \otimes \mathcal{S}_n$)

$$\alpha(\tilde{e}_i) = \sum_{j=1}^m \tilde{e}_j \otimes a_{i,j},$$

where by \tilde{e}_i we denote the image of the basis vector $e_i \in \mathcal{A}_n$ in \mathcal{A}_{n+1} . As $\tilde{e}_j = \sum_{r_j=1}^{l_j} e_{(j,s_j)}$,

$$\alpha(\tilde{e}_i) = \sum_{r_i=1}^{l_i} \alpha(e_{i,r_i}) = \sum_{r_i=1}^{l_i} \sum_{j=1}^m \sum_{s_j=1}^{l_j} e_{(j,s_j)} \otimes a_{(i,r_i),(j,s_j)}.$$

On the other hand we have

$$\alpha(\tilde{e}_i) = \sum_{j=1}^m \sum_{s_j=1}^{l_j} e_{(j,s_j)} \otimes a_{i,j},$$

and the comparison of the formulas above yields exactly (3.5.2). \square

One can deduce from the above lemma the exact structure of generators and relations between them for each \mathcal{S}_n associated with a commutative AF algebra. To be precise, if $\mathcal{A}_n = \mathbb{C}^{k_n}$ for some $k_n \in \mathbb{N}$, then the quantum isometry group \mathcal{S}_n is generated as a unital C^* -algebra by the family of self adjoint projections $\bigcup_{i=1}^n \{a_{\alpha_i, \beta_i} : \alpha_i, \beta_i = 1, \dots, k_i\}$ such that for each fixed $i = 1, \dots, n$ the family $\{a_{(\alpha_i, \beta_i)} : \alpha_i, \beta_i = 1, \dots, k_i\}$ satisfies the relations of \mathcal{QU}_{k_n} and the additional relations between $a_{(\alpha_i, \beta_i)}$ and $a_{(\alpha_{i+1}, \beta_{i+1})}$ for $i \in \{1, \dots, n-1\}$ are given by the formulas (3.5.2), after suitable reinterpretation of indices according to the multiplicities in the embedding of \mathbb{C}^{k_i} into $\mathbb{C}^{k_{i+1}}$.

Chapter 4

Quantum isometry groups for Rieffel deformed manifolds

In this chapter, we give a general scheme for computing $QISO^{\mathcal{L}}$ and \widetilde{QISO}_R^+ by proving that $QISO^{\mathcal{L}}$, (respectively \widetilde{QISO}_R^+) of a deformed noncommutative manifold coincides with (under reasonable assumptions) a similar deformation of $QISO^{\mathcal{L}}$, (respectively \widetilde{QISO}_R^+) of the original manifold.

4.1 Deformation of spectral triple

We recall from Chapter 1 the generalities of CQG s and Hopf algebras, in particular, the dense unital Hopf $*$ -subalgebra \mathcal{S}_0 of a CQG \mathcal{S} generated by the matrix elements of the irreducible unitary representations, the Sweedler convention for CQG action, as well as the convolutions $f \triangleleft c$, $c \triangleright f$ and $f \diamond g$ for functionals f, g on \mathcal{S} and c in \mathcal{S} . Moreover, given an action $\gamma : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{S}$ of the compact quantum group (\mathcal{S}, Δ) on a unital C^* -algebra \mathcal{A} , the dense, unital $*$ -subalgebra of \mathcal{A} on which γ becomes an action by the Hopf $*$ -algebra \mathcal{S}_0 is going to be denoted by \mathcal{A}_0 .

A word of caution: The algebra \mathcal{A}_0 should not be confused with the Rieffel deformed C^* algebra \mathcal{A}_J in the case $J = 0$, for which we simply write \mathcal{A} .

Let $(\mathcal{S}, \Delta_{\mathcal{S}})$ be a compact quantum group. We also adopt the convention of calling a vector space M an \mathcal{S} co-module if it is an algebraic \mathcal{S}_0 co-module in the sense of definition 1.2.13.

Before introducing the set up in which we are going to work, we prove the following well known fact for the sake of completeness.

Proposition 4.1.1. *Let E be a Banach space and G a second countable Lie group with a strongly continuous action α on E such that $\|\alpha_g(x)\| = \|x\|$ for all g in G and for all*

x in E . Then $E^\infty = \{e \in E : g \rightarrow \alpha_g(e) \text{ is } C^\infty\}$ is norm dense in \mathcal{A} .

Proof: For a compactly supported continuous function f on G and a in E , we will denote by $\alpha(f)(a)$ the norm convergent integral $\int_G f(h)\alpha_h(a)dh$ where dh denotes a left invariant Haar measure on G . Then, it can be seen that $\alpha_g(\alpha(f)a) = \int_G f(g^{-1}h)\alpha_h(a)dh$. Thus, for f in $C_c^\infty(G)$, $\alpha(f)(a)$ is in E^∞ . Now, for any $\epsilon > 0$, we choose a small enough neighbourhood U of identity of G , such that $\|\alpha_g(a) - a\| \leq \epsilon$ for all g in U . Next, we choose f in $C_c^\infty(G)$ with $f \geq 0$, $\int_G f dh = 1$ and $\text{supp}(f) \subseteq U$. Then,

$$\begin{aligned} & \|\alpha(f)(a) - a\| \\ &= \left\| \int_G f(g)\alpha_g(a)dg - a \int_G f(g)dg \right\| \\ &= \left\| \int_G f(g)(\alpha_g(a) - a)dg \right\| \\ &\leq \int_G f(g) \|\alpha_g(a) - a\| dg \\ &\leq \epsilon. \end{aligned}$$

This shows that E^∞ is dense in E . \square

Lemma 4.1.2. *Let \mathcal{A} be a C^* algebra with a strongly continuous action α of G as above. Then \mathcal{A}^∞ is closed under holomorphic functional calculus. Let ϕ be a positive linear map from \mathcal{A}^∞ to another C^* algebra \mathcal{B} . Then, for any self adjoint element x in \mathcal{A}^∞ , $\|\phi(x)\| \leq \|x\| \phi(1)$.*

Proof: The first fact is quite well known. We refer to [52] for a proof. For the second part, let x be a self adjoint element of \mathcal{A}^∞ . Then, $y = (1 + \epsilon)\|x\| - x$ is a positive and invertible element (since its spectrum does not contain zero) of \mathcal{A}^∞ , which being closed under holomorphic functional calculus, is closed under taking square root of an invertible element. Thus, $y^{\frac{1}{2}}$ belongs to \mathcal{A}^∞ and therefore $\phi(y) = \phi((y^{\frac{1}{2}})^* y^{\frac{1}{2}}) \geq 0$. This proves the Lemma. \square

Let $(\mathcal{A}, \mathbb{T}^n, \beta)$ be a C^* dynamical system, that is, \mathcal{A} is endowed with a strongly continuous action of \mathbb{T}^n by $*$ automorphisms. Moreover, $\pi_0 : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be a faithful representation, where \mathcal{H} is a separable Hilbert space.

Let \mathcal{A}^∞ be the smooth algebra corresponding to the \mathbb{T}^n action β .

Assume now that we are given a spectral triple $(\mathcal{A}^\infty, \pi_0, \mathcal{H}, D)$ of compact type. Suppose that D has eigenvalues $\{\lambda_0, \lambda_1, \dots\}$ and V_i denotes the (finite dimensional) eigenspace of λ_i and let \mathcal{S}_{00} denote the linear span of $\{V_i : i = 0, 1, 2, \dots\}$.

Suppose, furthermore, that there exists a compact abelian Lie group $\widetilde{\mathbb{T}^n}$, with a covering map $\gamma : \widetilde{\mathbb{T}^n} \rightarrow \mathbb{T}^n$. The Lie algebra of both \mathbb{T}^n and $\widetilde{\mathbb{T}^n}$ are isomorphic with

\mathbb{R}^n and we denote by e and \tilde{e} respectively the corresponding exponential maps, so that $e(u) = e(2\pi i u)$, $u \in \mathbb{R}^n$ and $\gamma(\tilde{e}(u)) = e(u)$. By a slight abuse of notation we shall denote the \mathbb{R}^n -action $\beta_{e(u)}$ by β_u .

Assumption:

There exists a strongly continuous unitary representation $V_{\tilde{g}}$, $\tilde{g} \in \widetilde{\mathbb{T}^n}$ of $\widetilde{\mathbb{T}^n}$ on \mathcal{H} such that

- (a) $V_{\tilde{g}}D = DV_{\tilde{g}}$ for all \tilde{g} ,
- (b) $V_{\tilde{g}}\pi_0(a)V_{\tilde{g}}^{-1} = \pi_0(\beta_g(a))$, where a belongs to \mathcal{A} , \tilde{g} belongs to $\widetilde{\mathbb{T}^n}$, and $g = \gamma(\tilde{g})$.

We shall now show that we can ‘deform’ the given spectral triple along the lines of [19]. For each J , the map $\pi_J : \mathcal{A}^\infty \rightarrow \text{Lin}(\mathcal{H}^\infty)$ (where \mathcal{H}^∞ is the smooth subspace corresponding to the representation V and $\text{Lin}(\mathcal{V})$ denotes the space of linear maps on a vector space \mathcal{V}) defined by

$$\pi_J(a)s \equiv a \times_J s := \int \int \beta_{Ju}(a)\tilde{\beta}_v(s)e(u.v)dudv$$

extends to a $*$ -representation of the C^* -algebra \mathcal{A}^∞ in $\mathcal{B}(\mathcal{H})$ where $\tilde{\beta}_v = V_{\tilde{e}(v)}$ (which clearly maps \mathcal{H}^∞ into \mathcal{H}^∞).

We can extend the action of \mathbb{T}^n on the C^* subalgebra \mathcal{A}_1 of $\mathcal{B}(\mathcal{H})$ generated by $\pi_0(\mathcal{A})$, $\{e^{itD} : t \in \mathbb{R}\}$ and elements of the form $\{[D, a] : a \in \mathcal{A}^\infty\}$ by $\beta_g(X) = V_{\tilde{g}}XV_{\tilde{g}}^{-1}$ for all X in \mathcal{A}_1 where by an abuse of notation, we denote the action by the same symbol β . Let \mathcal{A}_1^∞ denote the smooth vectors of \mathcal{A}_1 with respect to this action. We note that for all a in \mathcal{A}_1^∞ , $[D, a]$ belongs to \mathcal{A}_1^∞ .

Lemma 4.1.3. *β is a strongly continuous action (in the C^* -sense) of \mathbb{T}^n on \mathcal{A}_1 and hence for all X in \mathcal{A}_1^∞ , $\pi_J(X)$ defined by*

$$\pi_J(X)s = \int \int \beta_{Ju}(X)\tilde{\beta}_v(s)e(u.v)dudv$$

is a bounded operator.

Proof: We note that β is already strongly continuous on the C^* algebra generated by $\pi_0(\mathcal{A})$, $\{e^{itD} : t \in \mathbb{R}\}$. Thus it suffices to check the statement for elements of the form $[D, a]$ where a belongs to \mathcal{A}^∞ .

To this end, fix any one parameter subgroup g_t of \mathbb{T}^n such that g_t goes to the identity of \mathbb{T}^n as $t \rightarrow 0$. Let T'_t , \tilde{T}_t denote the group of normal $*$ -automorphisms on $\mathcal{B}(\mathcal{H})$ defined by $T'_t(X) = V_{g_t}XV_{g_t}^{-1}$ and $\tilde{T}_t(X) = e^{itD}Xe^{-itD}$. As V_{g_t} and D commute, so do the generators of T'_t and \tilde{T}_t . In particular, each of these generators leave the domain of the other invariant. Note also that \mathcal{A}^∞ is in the domain of the both the generators, and the generator of \tilde{T}_t is given by $[D, \cdot]$ there. Thus, for a in \mathcal{A}^∞ , we have

$a, [D, a]$ belong to $\text{Dom}(\Xi)$ (where Ξ is the generator of T'_t), and $\Xi([D, a]) = [D, \Xi(a)]$ belongs to $\mathcal{B}(\mathcal{H})$.

Using this, we obtain

$$\|T'_t([D, a]) - [D, a]\| = \int_0^t T'_s(\Xi([D, a]))ds \leq t \|\Xi([D, a])\|.$$

The required strong continuity follows from this. Then applying Proposition 1.3.5 to the C^* algebra \mathcal{A}_1 and the action β , we deduce that $\pi_J(X)$ is a bounded operator. \square

Lemma 4.1.4. *For each J , $(\mathcal{A}_J^\infty, \pi_J, \mathcal{H}, D)$ is a spectral triple, that is, $[D, \pi_J(a)]$ belongs to $\mathcal{B}(\mathcal{H})$ for all a in \mathcal{A}_J^∞ .*

Proof: $[D, \pi_J(a)](s) = D \int \int \beta_{Ju}(a) \widetilde{\beta}_v(s) e(u.v) dudv - \int \int \beta_{Ju}(a) \widetilde{\beta}_v(Ds) e(u.v) dudv$.

Using (1.3.1) and closability of D , we have

$$D \int \int \beta_{Ju}(a) \widetilde{\beta}_v(s) e(u.v) dudv = \int \int D(\beta_{Ju}(a) \widetilde{\beta}_v(s)) e(u.v) dudv.$$

As D commutes with V , the above expression equals

$$\int \int D(\beta_{Ju}(a) \widetilde{\beta}_v(s)) e(u.v) dudv - \int \int \beta_{Ju}(a) D \widetilde{\beta}_v(s) e(u.v) dudv.$$

So we have

$$\begin{aligned} [D, \pi_J(a)](s) &= \int \int [D, \beta_{Ju}(a)] \widetilde{\beta}_v(s) e(u.v) dudv \\ &= \int \int V_{Ju} [D, a] V_{Ju}^{-1} \widetilde{\beta}_v(s) e(u.v) dudv \\ &= \pi_J([D, a]), \end{aligned}$$

which is a bounded operator by Lemma 4.1.3. \square

4.2 Some preparatory results

In this section, we prove some preparatory results which will be needed in the next two sections. Let \mathbb{T}^n , $\widetilde{\mathbb{T}}^n, \beta, \widetilde{\beta}, \gamma$ be as in the previous subsection. By abuse of notation, we will use the symbols β and $\widetilde{\beta}$ for the corresponding comodule maps also. Let γ^*, γ_* be the canonical maps induced by γ from $C(\mathbb{T}^n) \rightarrow C(\widetilde{\mathbb{T}}^n)$ and $\text{Lie}(\widetilde{\mathbb{T}}^n) \rightarrow \text{Lie}(\mathbb{T}^n)$ respectively. Moreover, from now on, we will identify \mathcal{A}_J^∞ with $\pi_J(\mathcal{A}^\infty)$ and often write $\pi_0(a)$ simply as a .

Assumption

2a. Let $(\tilde{\mathcal{Q}}, \Delta)$ be a CQG and \mathcal{Q} a Woronowicz C^* subalgebra of $\tilde{\mathcal{Q}}$. Let there exist unital $*$ -subalgebra $\mathcal{A}_0 \subseteq \mathcal{A}$, which is norm dense in every \mathcal{A}_J , such that α is an action : $\alpha(\pi_0(\mathcal{A}_0)) \subseteq \pi_0(\mathcal{A}_0) \otimes \mathcal{Q}_0$. Let \mathcal{S}_0 be a vector subspace of \mathcal{H} (not necessarily closed) such that there is a map $\tilde{\alpha} : \mathcal{S}_0 \rightarrow \mathcal{S}_0 \otimes_{alg} \tilde{\mathcal{Q}}_0$ making it into an algebraic $\tilde{\mathcal{Q}}_0$ co module. Moreover, $(\text{id} \otimes \pi_{\tilde{\mathcal{Q}}})\tilde{\alpha} = \tilde{\beta}$.

2b. $C(\tilde{\mathbb{T}}^n)$ is a quantum subgroup of $\tilde{\mathcal{Q}}$, the quotient map being denoted by $\pi_{\tilde{\mathcal{Q}}}$.

2c. $\tilde{\alpha}(as) = \alpha(a)\tilde{\alpha}(s)$ for a in \mathcal{A}_0 , s in \mathcal{S}_0 .

We recall that we shall denote by η the canonical homomorphism from \mathbb{R}^n to \mathbb{T}^n given by $\eta(x_1, x_2, \dots, x_n) = (e(x_1), e(x_2), \dots, e(x_n))$. Moreover, we define $\Omega(u) := \text{ev}_{e(u)} \circ \pi_{\mathcal{Q}}$, $\tilde{\Omega}(u) := \text{ev}_{\tilde{e}(u)} \circ \pi_{\tilde{\mathcal{Q}}}$, for u in \mathbb{R}^n , where ev_x (respectively $\text{ev}_{\tilde{x}}$) denotes the state on $C(\mathbb{T}^n)$ (respectively, on $C(\tilde{\mathbb{T}}^n)$) obtained by evaluation of a function at the point x (respectively \tilde{x}).

We now make some observations.

Lemma 4.2.1. 1. From assumption **2c.**, it follows that $\text{ad}_{\tilde{\alpha}} = \alpha$.

$$2. (\text{id} \otimes \pi_{\tilde{\mathcal{Q}}})\text{ad}_{(\tilde{\alpha})} = \text{ad}_{\tilde{\beta}}.$$

$$3. \tilde{\beta}_x = (\text{id} \otimes \tilde{\Omega}(x))\tilde{\alpha}.$$

$$4. \beta_x = (\text{id} \otimes \Omega(x))\alpha.$$

5. $(\gamma^*)^{-1} \circ \pi_{\tilde{\mathcal{Q}}}$ is a surjective C^* homomorphism from \mathcal{Q} to $C(\mathbb{T}^n)$ identifying $C(\mathbb{T}^n)$ as a quantum subgroup of \mathcal{Q} .

Proof : By using (1.2.1), we have

$$\begin{aligned} \text{ad}_{\tilde{\alpha}}(a)s &= \tilde{\alpha}(a \otimes \text{id})\tilde{\alpha}^{-1}(s) \\ &= \tilde{\alpha}(a \otimes \text{id})(s_{(1)} \otimes \kappa(s_{(2)})) \\ &= \tilde{\alpha}(as_{(1)}) \otimes \kappa(s_{(2)}) \\ &= \alpha(a)\tilde{\alpha}(s_{(1)}) \otimes \kappa(s_{(2)}) \\ &= \alpha(a)(\tilde{\alpha} \otimes \text{id})(\text{id} \otimes \kappa)\tilde{\alpha}(s) \\ &= \alpha(a)(\tilde{\alpha} \otimes \text{id})(\tilde{\alpha}^{-1} \otimes \text{id})(s) \\ &= \alpha(a)s, \end{aligned}$$

where we have used Sweedler notations. This proves 1.

2. follows from 1. and the fact that $\pi_{\widetilde{\mathcal{Q}}}$ is a homomorphism.

$$\begin{aligned}\widetilde{\beta}_x(h) &= \widetilde{\beta}(h)(\widetilde{e}(x)) = (\text{id} \otimes \widetilde{\pi}_{\widetilde{\mathcal{Q}}})\widetilde{\alpha}(h)(\widetilde{e}(x)) = (\text{id} \otimes \text{ev}_{\widetilde{e}(x)}\widetilde{\pi}_{\widetilde{\mathcal{Q}}})\widetilde{\alpha}(h) \\ &= (\text{id} \otimes \widetilde{\Omega}(x))\widetilde{\alpha}(h).\end{aligned}$$

Therefore, $\widetilde{\beta}_x = (\text{id} \otimes \widetilde{\Omega}(x))\widetilde{\alpha}$. Similarly, 4. follows from 2.

We now prove 5. Let us denote by γ^* the dual map of γ , so that $\gamma^* : C(\mathbb{T}^n) \rightarrow C(\widetilde{\mathbb{T}}^n)$ is an injective C^* -homomorphism. It is quite clear that $(\text{id} \otimes \pi_{\widetilde{\mathcal{Q}}}) \circ \alpha(\mathcal{A}_0) \subseteq \text{Im}(\text{id} \otimes \gamma^*)$, hence it follows that $\pi_{\widetilde{\mathcal{Q}}}(\mathcal{Q}_0) \subseteq \text{Im}(\gamma^*)$. Thus, $\pi_{\mathcal{Q}} := (\gamma^*)^{-1} \circ \pi_{\widetilde{\mathcal{Q}}}$ is a surjective CQG morphism from \mathcal{Q} to $C(\mathbb{T}^n)$, which identifies $C(\mathbb{T}^n)$ as a quantum subgroup of \mathcal{Q} . \square

For a fixed J , we shall work with several multiplications on the vector space $\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{Q}_0$. We shall denote the counit and antipode of $\widetilde{\mathcal{Q}}_0$ by ϵ and κ respectively. Let us define the following operation :

$$x \odot y = \int_{\mathbb{R}^{4n}} e(-u.v)e(w.s)(\widetilde{\Omega}(-Ju) \triangleleft x \triangleright (\widetilde{\Omega}(Jw)))(\widetilde{\Omega}(-v) \triangleleft y \triangleright \widetilde{\Omega}(s))dudvdwds,$$

where x, y belong to $\widetilde{\mathcal{Q}}_0$. Then \odot is a bilinear maps, and will be seen to be associative multiplication later on.

We note that when x is in \mathcal{Q}_0 , f in $C(\mathbb{T}^n)$, $\gamma^*(f)(\widetilde{e}(u)) = f(\gamma(\widetilde{e}(u))) = f(e(\gamma_*(u))) = f(e(u))$ (as γ is a covering map). Using this, we have

$$\begin{aligned}(\widetilde{\Omega}(u) \otimes \text{id})\Delta(x) &= (ev_{\widetilde{e}(u)}\pi_{\widetilde{\mathcal{Q}}} \otimes \text{id})\Delta(x) \\ &= (ev_{\widetilde{e}(u)}\gamma^*\pi_{\mathcal{Q}} \otimes \text{id})\Delta(x) \\ &\quad (\text{as } \gamma^*\pi_{\mathcal{Q}} = \pi_{\widetilde{\mathcal{Q}}}) \\ &= (ev_{e(u)}\pi_{\mathcal{Q}} \otimes \text{id})\Delta(x) \\ &= (\Omega(u) \otimes \text{id})\Delta(x)\end{aligned}$$

and thus when x belongs to \mathcal{Q}_0 ,

$$(\widetilde{\Omega}(u) \otimes \text{id})\Delta(x) = (\Omega(u) \otimes \text{id})\Delta(x). \quad (4.2.1)$$

Moreover, we define bilinear maps \bullet, \bullet_J , by setting $(a \otimes x) \bullet (b \otimes y) := ab \otimes x \odot y$, $(a \otimes x) \bullet_J (b \otimes y) := (a \times_J b) \otimes (x \odot y)$, for a, b in \mathcal{A}_0 , x, y in $\widetilde{\mathcal{Q}}_0$.

Lemma 4.2.2. *For x in $\widetilde{\mathcal{Q}}_0$, we have*

$$\widetilde{\Omega}(u) \triangleleft (\widetilde{\Omega}(v) \triangleleft x) = (\widetilde{\Omega}(u) \diamond \widetilde{\Omega}(v)) \triangleleft x.$$

For x in \mathcal{Q}_0 , we have

$$\Omega(u) \triangleleft (\Omega(v) \triangleleft x) = (\Omega(u) \diamond \Omega(v)) \triangleleft x.$$

Proof : We will denote by $\Delta_{\widetilde{\mathbb{T}^n}}$ the coproduct on $C(\widetilde{\mathbb{T}^n})$, hence, we have

$$(\pi_{\widetilde{\mathcal{Q}}} \otimes \pi_{\widetilde{\mathcal{Q}}})\Delta = \Delta_{\widetilde{\mathbb{T}^n}}\pi_{\widetilde{\mathcal{Q}}} \quad (4.2.2)$$

Moreover, we note that as \mathbb{T}^n is a commutative group, $f \diamond g = g \diamond f$ for any two functionals f and g on $C(\mathbb{T}^n)$.

$$\begin{aligned} & \widetilde{\Omega}(u) \triangleleft (\widetilde{\Omega}(v) \triangleleft x) \\ &= (\widetilde{\Omega}(u) \otimes \text{id})\Delta(\widetilde{\Omega}(v) \triangleleft x) \\ &= (\widetilde{\Omega}(u) \otimes \text{id})\Delta(\widetilde{\Omega}(v)(x_{(1)}) \cdot x_{(2)}) \\ &= (\widetilde{\Omega}(u) \otimes \text{id})\Delta(x_{(2)})\widetilde{\Omega}(v)(x_{(1)}) \\ &= (\widetilde{\Omega}(v) \otimes \widetilde{\Omega}(u) \otimes \text{id})(x_{(1)} \otimes x_{(2)(1)} \otimes x_{(2)(2)}) \\ &= (\widetilde{\Omega}(v) \otimes \widetilde{\Omega}(u) \otimes \text{id})((\text{id} \otimes \Delta)\Delta(x)) \\ &= (\widetilde{\Omega}(v) \otimes \widetilde{\Omega}(u) \otimes \text{id})((\Delta \otimes \text{id})\Delta(x)) \\ &= (\widetilde{\Omega}(v) \otimes \widetilde{\Omega}(u))\Delta(x_{(1)}) \otimes x_{(2)} \\ &= (ev_{\eta(v)} \otimes ev_{\eta(u)})(\pi_{\widetilde{\mathcal{Q}}} \otimes \pi_{\widetilde{\mathcal{Q}}})\Delta(x_{(1)}) \otimes x_{(2)}, \end{aligned}$$

which by (4.2.2) equals

$$\begin{aligned} & (ev_{\eta(v)} \otimes ev_{\eta(u)})\Delta_{\mathbb{T}^n}\pi_{\widetilde{\mathcal{Q}}}(x_{(1)}) \otimes x_{(2)} \\ &= (ev_{\eta(v)} \diamond ev_{\eta(u)})(\pi_{\widetilde{\mathcal{Q}}}(x_{(1)}) \otimes x_{(2)}) \\ &= (ev_{\eta(u)} \diamond ev_{\eta(v)})(\pi_{\widetilde{\mathcal{Q}}}(x_{(1)}) \otimes x_{(2)}) \\ &= (\widetilde{\Omega}(u) \otimes \widetilde{\Omega}(v))\Delta(x_{(1)}) \otimes x_{(2)} \\ &= (\widetilde{\Omega}(u) \diamond \widetilde{\Omega}(v))(x_{(1)}) \otimes x_{(2)} \\ &= ((\widetilde{\Omega}(u) \diamond \widetilde{\Omega}(v)) \otimes \text{id})\Delta(x) \\ &= (\widetilde{\Omega}(u) \diamond \widetilde{\Omega}(v)) \triangleleft x. \end{aligned}$$

The second part follows from this and using (4.2.1).

□

Lemma 4.2.3. *The map \odot satisfies*

$$\int_{\mathbb{R}^{2n}} (\tilde{\Omega}(Ju) \triangleleft x) \odot (\tilde{\Omega}(v) \triangleleft y) e(u.v) dudv = \int_{\mathbb{R}^{2n}} (x \triangleright (\tilde{\Omega}(Ju))) (y \triangleright \tilde{\Omega}(v)) e(u.v) dudv,$$

for x, y in $\widetilde{\mathcal{Q}}_0$. When x, y are in \mathcal{Q}_0 , we have

$$\int_{\mathbb{R}^{2n}} (\Omega(Ju) \triangleleft x) \odot (\Omega(v) \triangleleft y) e(u.v) dudv = \int_{\mathbb{R}^{2n}} (x \triangleright (\Omega(Ju))) (y \triangleright \Omega(v)) e(u.v) dudv.$$

Proof : The expression in the left hand side equals

$$\begin{aligned} & \int (\tilde{\Omega}(Ju') \triangleleft x) \odot (\tilde{\Omega}(v') \triangleleft y) e(u'.v') du' dv' \\ &= \int_{\mathbb{R}^{2n}} \left\{ \int_{\mathbb{R}^{4n}} e(-u.v) e(w.s) (\tilde{\Omega}(-Ju) \triangleleft (\tilde{\Omega}(Ju') \triangleleft x) \triangleright \tilde{\Omega}(Jw)) \right. \\ & \quad \left. (\tilde{\Omega}(-v) \triangleleft (\tilde{\Omega}(v') \triangleleft y) \triangleright \tilde{\Omega}(s)) dudvdwds \right\} e(u'.v') du' dv' \\ &= \int_{\mathbb{R}^{6n}} (\tilde{\Omega}(J(u' - u)) \triangleleft x) \triangleright \tilde{\Omega}(Jw) (\tilde{\Omega}(v' - v) \triangleleft y \triangleright \tilde{\Omega}(s)) \\ & \quad e(u'.v') e(-u.v) e(w.s) dudvdwds du' dv' \\ &= \int_{\mathbb{R}^{2n}} e(w.s) dw ds \left\{ \int_{\mathbb{R}^{4n}} e(u'.v') e(-u.v) dudv du' dv' \right. \\ & \quad \left. (\tilde{\Omega}(J(u' - u)) \triangleleft x_w) (\tilde{\Omega}(v' - v) \triangleleft y_s) \right\}, \end{aligned}$$

where $x_w = x \triangleright \tilde{\Omega}(Jw)$, $y_s = y \triangleright \tilde{\Omega}(s)$.

The proof of the lemma will be complete if we show

$$\int_{\mathbb{R}^{4n}} e(u'.v') e(-u.v) (\tilde{\Omega}(J(u' - u)) \triangleleft x_w) (\tilde{\Omega}(v' - v) \triangleleft y_s) dudv du' dv' = x_w . y_s.$$

By changing variable in the above integral, with $z = u' - u$, $t = v' - v$, it becomes

$$\begin{aligned} & \int_{\mathbb{R}^{4n}} e(-u.v) e((u+z).(v+t)) \phi(z, t) dudvdzdt \\ &= \int_{\mathbb{R}^{4n}} \phi(z, t) e(u.t + z.v) e(z.t) dudvdzdt, \text{ where} \end{aligned}$$

$$\phi(z, t) = (\tilde{\Omega}(J(z)) \triangleleft x_w) (\tilde{\Omega}(t) \triangleleft y_s).$$

By taking $(z, t) = X$, $(v, u) = Y$, and $F(X) = \phi(z, t) e(z.t)$, the integral can be written as

$$\begin{aligned} & \int \int F(X) e(X.Y) dX dY \\ &= F(0) \text{ (by Proposition 1.3.1)} \\ &= (\tilde{\Omega}(J(0)) \triangleleft x_w) (\tilde{\Omega}(0) \triangleleft y_s) \\ &= x_w . y_s, \end{aligned}$$

since

$$\tilde{\Omega}(J(0)) \triangleleft x_w = (ev_{\eta(0)} \pi_{\tilde{Q}} \otimes \text{id}) \Delta(x_w) = (\epsilon_{\tilde{\mathbb{T}}^n} \circ \pi_{\tilde{Q}} \otimes \text{id}) \Delta(x_w) = (\epsilon \otimes \text{id}) \Delta(x_w) = x_w$$

and similarly $\tilde{\Omega}(0) \triangleleft y_s = y_s$, where $\epsilon_{\tilde{\mathbb{T}}^n}$ denotes the counit of the quantum group $C(\tilde{\mathbb{T}}^n)$.

This proves the claim and hence the first part of the Lemma. The second part follows from this and (4.2.1). \square

Lemma 4.2.4. *We have for a in \mathcal{A}_0 , s in \mathcal{S}_0 ,*

$$\tilde{\alpha}(\tilde{\beta}_u(s)) = s_{(1)} \otimes (\text{id} \otimes \tilde{\Omega}(u))(\Delta(s_{(2)})), \quad (4.2.3)$$

$$\alpha(\beta_u(a)) = a_{(1)} \otimes (\text{id} \otimes \Omega(u))(\Delta(a_{(2)})). \quad (4.2.4)$$

Proof : $\tilde{\beta}_u = (\text{id} \otimes ev_u \circ \tilde{\pi})\tilde{\alpha}$. We have

$$\begin{aligned} \tilde{\beta}_u(s) &= (\text{id} \otimes \tilde{\Omega}(u))\tilde{\alpha}(s) \\ &= (\text{id} \otimes \tilde{\Omega}(u))(s_{(1)} \otimes s_{(2)}) \\ &= s_{(1)}(\tilde{\Omega}(u))(s_{(2)}). \end{aligned}$$

This gives,

$$\begin{aligned} \tilde{\alpha}(\tilde{\beta}_u(s)) &= \tilde{\alpha}(s_{(1)})\tilde{\Omega}(u)(s_{(2)}) \\ &= (\text{id} \otimes \text{id} \otimes \tilde{\Omega}(u))(\tilde{\alpha}(s_{(1)}) \otimes s_{(2)}) \\ &= (\text{id} \otimes \text{id} \otimes \tilde{\Omega}(u))((\tilde{\alpha} \otimes \text{id})\tilde{\alpha}(s)) \\ &= (\text{id} \otimes \text{id} \otimes \tilde{\Omega}(u))((\text{id} \otimes \Delta)\tilde{\alpha}(s)) \\ &= s_{(1)} \otimes (\text{id} \otimes \tilde{\Omega}(u))\Delta(s_{(2)}). \end{aligned}$$

Proceeding in a similar way, we obtain $\beta_u(a) = a_{(1)}(\Omega(u))(a_{(2)})$ for all a in \mathcal{A}_0 and hence $\alpha(\beta_u(a)) = a_{(1)} \otimes (\text{id} \otimes \Omega(u))(\Delta(a_{(2)}))$ for all a in \mathcal{A}_0 . \square

Lemma 4.2.5. *For all s in \mathcal{S}_0 , a in \mathcal{A}_0 , we have*

$$\tilde{\alpha}(a \times_J s) = a_{(1)}s_{(1)} \otimes \left(\int \int (a_{(2)} \triangleright \Omega(Ju))(s_{(2)} \triangleright \tilde{\Omega}(v))e(u.v)dudv \right). \quad (4.2.5)$$

For a, b in \mathcal{A}_0 , we have

$$\alpha(a \times_J b) = a_{(1)} b_{(1)} \otimes \left(\int \int (a_{(2)} \triangleright \Omega(Ju))(b_{(2)} \triangleright \Omega(v)) e(u.v) du dv \right). \quad (4.2.6)$$

Proof : Using the notations and definitions in section 1.3, we note that for any $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ belonging to $\mathcal{B}(\mathbb{R}^2)$ and fixed x in E (where E is a Banach algebra), the function $F(u, v) = x f(u, v)$ belongs to $\mathcal{B}^E(\mathbb{R}^2)$ and we have

$$\begin{aligned} & x \left(\int \int f(u, v) e(u.v) du dv \right) \\ &= x \left(\lim_L \sum_{p \in L} \int \int (f \phi_p)(u, v) e(u.v) du dv \right) \\ &= \lim_L \sum_{p \in L} \int \int x (f \phi_p)(u, v) e(u.v) du dv \\ &= \int \int x f(u, v) e(u.v) du dv. \end{aligned}$$

Then,

$$\begin{aligned} & \tilde{\alpha}(a \times_J s) \\ &= \tilde{\alpha} \left(\int \int \beta_{Ju}(a) \tilde{\beta}_v(s) e(u.v) du dv \right) \\ &= \tilde{\alpha} \left(\int \int a_{(1)} (\Omega(Ju)) (a_{(2)}) s_{(1)} (\tilde{\Omega}(v)) (s_{(2)}) e(u.v) du dv \right) \\ &= \tilde{\alpha}((a_{(1)} s_{(1)}) \int \int (\Omega(Ju)) (a_{(2)}) (\tilde{\Omega}(v)) (s_{(2)}) e(u.v) du dv) \\ &= \alpha(a_{(1)}) \tilde{\alpha}(s_{(1)}) \int \int (\Omega(Ju)) (a_{(2)}) (\tilde{\Omega}(v)) (s_{(2)}) e(u.v) du dv \\ &\quad \text{(by assumption 2.c)} \\ &= \int \int \alpha(a_{(1)}) (\Omega(Ju)) (a_{(2)}) \tilde{\alpha}(s_{(1)}) (\tilde{\Omega}(v)) (s_{(2)}) e(u.v) du dv \\ &= \int \int \alpha(a_{(1)} \Omega(Ju) (a_{(2)})) \tilde{\alpha}(s_{(1)} \tilde{\Omega}(v) (s_{(2)})) e(u.v) du dv \\ &= \int \int \alpha(\beta_{Ju}(a)) \tilde{\alpha}(\tilde{\beta}_v(s)) e(u.v) du dv \\ &= \int \int (a_{(1)} \otimes (\text{id} \otimes \Omega(Ju)) (\Delta(a_{(2)}))) (s_{(1)} \otimes (\text{id} \otimes \tilde{\Omega}(v))) (\Delta(s_{(2)})) \\ &\quad e(u.v) du dv \\ &\quad \text{(using Lemma 4.2.4)} \\ &= a_{(1)} s_{(1)} \otimes \int \int (a_{(2)} \triangleright \Omega(Ju)) (s_{(2)} \triangleright \tilde{\Omega}(v)) e(u.v) du dv. \end{aligned}$$

□

Lemma 4.2.6. For s in \mathcal{S}_0 , a in \mathcal{A}_0 ,

$$\alpha(a) \bullet_J \tilde{\alpha}(s) = a_{(1)} s_{(1)} \otimes \left(\int \int (\Omega(Ju) \triangleleft a_{(2)}) \odot (\tilde{\Omega}(v) \triangleleft s_{(2)}) e(u.v) dudv \right). \quad (4.2.7)$$

For a, b in \mathcal{A}_0 ,

$$\alpha(a) \bullet_J \alpha(b) = a_{(1)} b_{(1)} \otimes \left\{ \int \int (\Omega(Ju) \triangleleft a_{(2)}) \odot (\Omega(v) \triangleleft b_{(2)}) e(u.v) dudv \right\}. \quad (4.2.8)$$

Proof : We have

$$\begin{aligned} \alpha(a) \bullet_J \tilde{\alpha}(s) &= (a_{(1)} \otimes a_{(2)}) \bullet_J (s_{(1)} \otimes s_{(2)}) \\ &= a_{(1)} \times_J s_{(1)} \otimes (a_{(2)} \odot s_{(2)}) \\ &= \int \int \beta_{Ju}(a_{(1)}) \tilde{\beta}_v(s_{(1)}) e(u.v) dudv \otimes (a_{(2)} \odot s_{(2)}). \end{aligned}$$

Let ϵ be the counit of $\tilde{\mathcal{Q}}$. So we have $(\text{id} \otimes \epsilon)\alpha = \text{id}$ and $(\text{id} \otimes \epsilon)\tilde{\alpha} = \text{id}$. This gives,

$$\begin{aligned} \alpha(a) \bullet_J \tilde{\alpha}(s) &= \int \int (\text{id} \otimes \epsilon) \alpha(\beta_{Ju}(a_{(1)})) (\text{id} \otimes \epsilon) \tilde{\alpha}(\tilde{\beta}_v(s_{(1)})) e(u.v) dudv \otimes (a_{(2)} \odot s_{(2)}). \end{aligned}$$

Note that by Lemma 4.2.4, $\int \int (\text{id} \otimes \epsilon) (\alpha(\beta_{Ju}(a_{(1)})) (\text{id} \otimes \epsilon) (\tilde{\alpha}(\tilde{\beta}_v(s_{(1)}))) e(u.v) dudv$
 $= \int \int (\text{id} \otimes \epsilon) (a_{(1)(1)} \otimes (\text{id} \otimes \Omega(Ju)) (\Delta(a_{(1)(2)}))) (\text{id} \otimes \epsilon) (s_{(1)(1)} \otimes (\text{id} \otimes \tilde{\Omega}(v)) (\Delta(s_{(1)(2)}))) e(u.v) dudv$
 $= \int \int (\text{id} \otimes \epsilon) (a_{(1)(1)} \otimes (a_{(1)(2)} \triangleright \Omega(Ju)) (\text{id} \otimes \epsilon) (s_{(1)(1)} \otimes (s_{(1)(2)} \triangleright \Omega(v)))) e(u.v) dudv$
 $= \int \int a_{(1)(1)} s_{(1)(1)} \epsilon(a_{(1)(2)} \triangleright \Omega(Ju)) \epsilon(s_{(1)(2)} \triangleright \tilde{\Omega}(v)) e(u.v) dudv.$

Using the fact that $f \diamond \epsilon = \epsilon \diamond f = f$ for any functional on \mathcal{Q}_0 , one has $\epsilon(a_{(1)(2)} \triangleright \Omega(Ju)) = \Omega(Ju)(a_{(1)(2)})$ and $\epsilon(s_{(1)(2)} \triangleright \tilde{\Omega}(v)) = \tilde{\Omega}(v)(s_{(1)(2)})$, from which it follows that

$$\begin{aligned} \alpha(a) \bullet_J \tilde{\alpha}(s) &= a_{(1)(1)} s_{(1)(1)} \int \int \Omega(Ju)(a_{(1)(2)}) \tilde{\Omega}(v)(s_{(1)(2)}) e(u.v) dudv \otimes (a_{(2)} \odot s_{(2)}) \\ &= \int \int (\text{id} \otimes \Omega(Ju) \otimes \text{id}) (a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)}) \bullet (\text{id} \otimes \tilde{\Omega}(v) \otimes \text{id}) (s_{(1)(1)} \otimes s_{(1)(2)} \otimes s_{(2)}) e(u.v) dudv \end{aligned}$$

$$\begin{aligned}
&= \int \int (\text{id} \otimes \Omega(Ju) \otimes \text{id})(a_{(1)} \otimes \Delta(a_{(2)})) \bullet (\text{id} \otimes (\tilde{\Omega}(v) \otimes \text{id}))(s_{(1)} \otimes \Delta(s_{(2)})) \\
&\quad e(u.v) dudv \\
&= \int \int \{a_{(1)} \otimes (\Omega(Ju) \otimes \text{id})\Delta(a_{(2)})\} \bullet \{s_{(1)} \otimes (\tilde{\Omega}(v) \otimes \text{id})\Delta(s_{(2)})\} e(u.v) dudv \\
&= a_{(1)}s_{(1)} \otimes \int \int ((\tilde{\Omega}(Ju) \otimes \text{id})\Delta(a_{(2)})) \odot (\tilde{\Omega}(v) \otimes \text{id})\Delta(s_{(2)}) e(u.v) dudv \\
&\quad (\text{ by (4.2.1) }) \\
&= a_{(1)}s_{(1)} \otimes \int \int (\tilde{\Omega}(Ju) \triangleleft a_{(2)}) \odot (\tilde{\Omega}(v) \triangleleft s_{(2)}) e(u.v) dudv,
\end{aligned}$$

where we have used the relation $(\alpha \otimes \text{id})\alpha = (\text{id} \otimes \Delta)\alpha$ to get $a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes \Delta(a_{(2)})$ and similarly $s_{(1)(1)} \otimes s_{(1)(2)} \otimes s_{(2)} = s_{(1)} \otimes \Delta(s_{(2)})$. \square

Combining Lemma 4.2.3, Lemma 4.2.5 and Lemma 4.2.6, we conclude the following.

Lemma 4.2.7. *For a in \mathcal{A}_0 , s in \mathcal{S}_0 , we have*

$$\alpha(a) \bullet_J \tilde{\alpha}(s) = \tilde{\alpha}(a \times_J s). \quad (4.2.9)$$

For a, b in \mathcal{A}_0 , we have

$$\alpha(a) \bullet_J \alpha(b) = \alpha(a \times_J b). \quad (4.2.10)$$

We shall now identify \odot with the multiplication of a Rieffel-type deformation of $\mathcal{Q}(\tilde{\mathcal{Q}})$. We discuss the case for $\tilde{\mathcal{Q}}$, that of \mathcal{Q} being similar. Since $\tilde{\mathcal{Q}}$ has a quantum subgroup isomorphic with $C(\mathbb{T}^n)$, we can consider the following canonical action χ of \mathbb{R}^{2n} on $\tilde{\mathcal{Q}}$ (as in (1.3.8)) given by

$$\chi_{(s,u)} = (\tilde{\Omega}(-s) \otimes \text{id})\Delta(\text{id} \otimes \tilde{\Omega}(u))\Delta.$$

Now, let $\tilde{J} := -J \oplus J$, which is a skew-symmetric $2n \times 2n$ real matrix, so one can deform $\tilde{\mathcal{Q}}$ by defining the product of x and y (x, y belonging to $\tilde{\mathcal{Q}}_0$, say) to be the following:

$$\int \int \chi_{\tilde{J}(u,w)}(x) \chi_{v,s}(y) e((u,w).(v,s)) d(u,w) d(v,s).$$

We claim that this is nothing but \odot introduced before.

Lemma 4.2.8.

$$x \odot y = x \times_{\tilde{J}} y \quad \text{for all } x, y \in \tilde{\mathcal{Q}}_0.$$

Proof : Let us first observe that

$$\begin{aligned} \chi_{\tilde{J}(u,w)}(x) &= (\tilde{\Omega}(Ju) \otimes \text{id})\Delta(\text{id} \otimes \tilde{\Omega}(Jw))\Delta(x) \\ &= \tilde{\Omega}(Ju) \triangleleft x \triangleright \tilde{\Omega}(Jw), \end{aligned}$$

and similarly $\chi_{(v,s)}(y) = \tilde{\Omega}(-v) \triangleleft y \triangleright \tilde{\Omega}(s)$.

Thus, we have

$$\begin{aligned} x \odot y &= \int_{\mathbb{R}^{4n}} (\tilde{\Omega}(-Ju) \triangleleft x \triangleright \tilde{\Omega}(Jw))(\tilde{\Omega}(-v) \triangleleft y \triangleright \tilde{\Omega}(s))e(-u.v)e(w.s)dudvdwds \\ &= \int_{\mathbb{R}^{4n}} (\tilde{\Omega}(Ju') \triangleleft x \triangleright \tilde{\Omega}(Jw))(\tilde{\Omega}(-v) \triangleleft y \triangleright \tilde{\Omega}(s))e(u'.v)e(w.s)du'dvdwds \\ &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \chi_{\tilde{J}(u,w)}(x)\chi_{(v,s)}(y)e((u,w).(v,s))d(u,w)d(v,s), \end{aligned}$$

which proves the claim. \square

Let us denote by $\tilde{\mathcal{Q}}_{\tilde{J}}$ ($\mathcal{Q}_{\tilde{J}}$) the C^* algebra obtained from $\tilde{\mathcal{Q}}$ (\mathcal{Q}) by the Rieffel deformation w.r.t. the matrix \tilde{J} described above. We recall from subsection 1.3.1 that the coproduct Δ on $\tilde{\mathcal{Q}}_0$ (\mathcal{Q}_0) extends to a coproduct for the deformed algebra as well and $(\tilde{\mathcal{Q}}_{\tilde{J}}, \Delta)$ ($(\mathcal{Q}_{\tilde{J}}, \Delta)$) is a compact quantum group.

4.3 \widetilde{QISO}_R^+ of a Rieffel deformed noncommutative manifold

4.3.1 Derivation of the result

In this subsection, our set up is as in section 4.1 so that we have spectral triples on \mathcal{A}_J^∞ for each J .

Lemma 4.3.1. *Suppose that $(\tilde{\mathcal{Q}}, U)$ belongs to $\text{Obj}(\mathbf{Q}(\mathcal{A}, \mathcal{H}, D))$, and there exists a unital $*$ -subalgebra $\mathcal{A}_0 \subseteq \mathcal{A}$ which is norm dense in every \mathcal{A}_J such that*

$\alpha_U(\pi_0(\mathcal{A}_0)) \subseteq \pi_0(\mathcal{A}_0) \otimes_{\text{alg}} \mathcal{Q}_0$, where $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$ is the smallest Woronowicz C^ subalgebra such that $\alpha_U(\mathcal{A}_0) \subseteq \pi_0(\mathcal{A}_0) \otimes \mathcal{Q}$, and \mathcal{Q}_0 is the Hopf $*$ -algebra obtained by matrix coefficients of irreducible unitary (co)-representations of \mathcal{Q} . Also, let $S_0 = \text{span}\{as : a \in \mathcal{A}_0, s \in S_{00}\}$, Then we have the following:*

$$(a) \ U(S_0) \subseteq \mathcal{S}_0 \otimes_{\text{alg}} \tilde{\mathcal{Q}}_0.$$

(b) $\tilde{\alpha} := U|_{S_0} : S_0 \rightarrow S_0 \otimes_{\text{alg}} \tilde{\mathcal{Q}}_0$ makes S_0 an algebraic $\tilde{\mathcal{Q}}_0$ co-module, satisfying

$$\tilde{\alpha}(\pi_0(a)s) = \alpha_U(a)\tilde{\alpha}(s) \text{ for all } a \text{ in } \mathcal{A}_0, s \text{ in } S_0.$$

Moreover, if $(C(\mathbb{T}^n), V)$ is a sub object of $\tilde{\mathcal{Q}}$ in $\mathbf{Q}(\mathcal{A}, \mathcal{H}, D)$ such that $V(\cdot \otimes \text{id})V^* = \beta$, then $C(\mathbb{T}^n)$ is a quantum subgroup of \mathcal{Q} .

Proof: U commutes with D and hence preserves the eigenspaces of D which shows that U preserves S_{00} . Then, $\tilde{U}(as \otimes 1) = \alpha(a)\tilde{U}(s \otimes 1) \subseteq (\mathcal{A}_0 \otimes \mathcal{Q}_0)(S_{00} \otimes \tilde{\mathcal{Q}}_0) \subseteq S_0 \otimes \tilde{\mathcal{Q}}_0$. Thus, the first assertion follows.

The second assertion follows from the definition of $\tilde{\alpha}$ and α_U . The third assertion follows as in Lemma 4.2.1. \square

Remark 4.3.2. From the definitions of \mathcal{A}_0 and S_0 , it follows that

- (i) $\pi_0(\mathcal{A}_0)S_0 \subseteq S_0$,
- (ii) $\beta_g(\mathcal{A}_0) \subseteq \mathcal{A}_0$ for all g .

Let us now fix the object $(\tilde{\mathcal{Q}}, U)$ as in the statement of Lemma 4.3.1. We recall that using the identification of \mathcal{Q}_0 as a common vector-subspace of all $\mathcal{Q}_{\tilde{J}}$, we shall sometimes denote this identification map from \mathcal{Q}_0 to $\mathcal{Q}_{\tilde{J}}$ by ρ_J .

Let us consider the finite dimensional unitary representations $U^{(i)} := U|_{V_i}$, where V_i is the eigenspace of D corresponding to the eigenvalue λ_i . By Corollary 1.3.14, we can view $U^{(i)}$ as a unitary representation of $\mathcal{Q}_{\tilde{J}}$ as well, and let us denote it by $U_J^{(i)}$. In this way, we obtain a unitary representation U_J on the Hilbert space \mathcal{H} , which is the closed linear span of all the V_i 's. It is obvious from the construction (and the fact that the linear span of V_i 's, that is S_0 , is a core for D) that $U_J D = (D \otimes I)U_J$. Let $\alpha_J := \alpha_{U_J}$. With this, we have the following:

Lemma 4.3.3. For a in \mathcal{A}_0 , we have $\alpha_J(a) = (\alpha(a))_J \equiv (\pi_J \otimes \rho_J)(\alpha(a))$, hence in particular, for every state ϕ on $\mathcal{Q}_{\tilde{J}}$, $(\text{id} \otimes \phi) \circ \alpha_J(\mathcal{A}_J) \subseteq \mathcal{A}_J''$.

Using the equation (4.2.9), we have for all s in S_0, a in \mathcal{A}_0 ,

$$\begin{aligned} & \alpha_J(a)U_J(s) \\ &= U_J(\pi_J(a)s) \\ &= \tilde{\alpha}(a \times_J s) \\ &= \alpha(a) \bullet_J \tilde{\alpha}(s) \\ &= (\alpha(a))_J U_J(s), \end{aligned}$$

from which we conclude by the density of \mathcal{S}_0 in \mathcal{H} that $\alpha_J(a) = (\alpha(a))_J$ belongs to $\pi_J(\mathcal{A}_0) \otimes \mathcal{Q}_{\tilde{J}}$. The lemma now follows using the norm-density of \mathcal{A}_0 in \mathcal{A}_J . \square

Corollary 4.3.4. $(\tilde{\mathcal{Q}}_{\tilde{J}}, U_J)$ is an orientation preserving isometric action on the spectral triple $(\mathcal{A}_J^\infty, \mathcal{H}, D)$.

We shall now show that if we fix a ‘volume-form’ in terms of an R -twisted structure, then the ‘deformed’ action α_J preserves it.

Lemma 4.3.5. Suppose, in addition to the set-up already assumed, that there is an invertible positive operator R on \mathcal{H} such that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is an R -twisted Θ -summable spectral triple, and let τ_R be the corresponding ‘volume form’. Assume that α_U preserves the functional τ_R . Then the action α_{U_J} preserves τ_R too.

Proof : Let the (finite dimensional) eigenspace corresponding to the eigenvalue λ_n of D be V_n . As U commutes with D , there exists subspaces $V_{n,k}$ of V_n and an orthonormal basis $\{e_j^{n,k}\}_j$ for $V_{n,k}$ such that the restriction of U to $V_{n,k}$ is irreducible. Write $\tilde{U}(e_j^{n,k} \otimes 1) = \sum_i e_i^{n,k} \otimes t_{i,j}^n$. Then, $\tilde{U}^*(e_j^{n,k}) = \sum_i e_i^{n,k} \otimes t_{j,i}^{n*}$.

Then \mathcal{H} will be decomposed as $\mathcal{H} = \bigoplus_{n \geq 1, k} V_{n,k}$.

Let $R(e_j^{n,i}) = \sum_{s,t} F_n(i, j, s, t) e_t^{n,s}$.

By hypothesis, $\tilde{U}(\cdot \otimes \text{id})\tilde{U}^*$ preserves the functional $\tau_R(\cdot) = \text{Tr}(R \cdot)$ on \mathcal{E}_D where \mathcal{E}_D is as in Proposition 3.2.7, that is the weakly dense $*$ subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the rank one operators $|\xi\rangle\langle\eta|$ where ξ, η are eigenvectors of D . Thus, $(\tau_R \otimes \text{id})(\tilde{U}(X \otimes \text{id})\tilde{U}^*) = \tau_R(X) \cdot 1_Q$ for all X in \mathcal{E}_D .

Then, for a in \mathcal{E}_D , we have:

$$\begin{aligned}
& (\tau_R \otimes h)(\tilde{U}_J(a \otimes 1)\tilde{U}_J^*) \\
&= \sum_{n,i,j} \left\langle e_j^{n,i} \otimes 1, \tilde{U}_J(a \otimes 1)\tilde{U}_J^*(R e_j^{n,i} \otimes 1) \right\rangle \\
&= \sum_{n,i,j,s,t} \left\langle \tilde{U}_J^*(e_j^{n,i} \otimes 1), (a \otimes 1)\tilde{U}_J^*(F_n(i, j, s, t) e_t^{n,s} \otimes 1) \right\rangle \\
&= \sum_{n,i,j,s,t,k,l} F_n(i, j, s, t) \left\langle e_k^{n,i} \otimes (t_{j,k}^n)^*, (a \otimes 1)(e_l^{n,s} \otimes (t_{t,l}^n)^*) \right\rangle \\
&= \sum_{n,i,j,s,t,k,l} F_n(i, j, s, t) \left\langle e_k^{n,i}, a e_l^{n,s} \right\rangle h_J((t_{j,k}^n) \times_J (t_{t,l}^n)^*) \\
&= \sum_{n,i,j,s,t,k,l} F_n(i, j, s, t) \left\langle e_k^{n,i}, a e_l^{n,s} \right\rangle h_0(t_{j,k}^n t_{t,l}^{n*}) \\
&= (\tau_R \otimes h)(\tilde{U}(a \otimes 1)\tilde{U}^*) \\
&= \tau_R(a) \cdot 1
\end{aligned}$$

where $h_J((t_{j,k}^n) \times_J (t_{t,l}^n)^*) = h_0(t_{j,k}^n t_{t,l}^{n*})$ as deduced by using Lemma 1.3.10.

Thus $(\tau_R \otimes h)\widetilde{U}_J(a \otimes \text{id})\widetilde{U}_J^* = \tau_R(a).1$

Let $(\tau_R \otimes h)\widetilde{U}_J(X \otimes \text{id})\widetilde{U}_J^* = (\tau_R * h)(X)$. As $\widetilde{U}_J(\cdot \otimes \text{id})\widetilde{U}_J^*$ keeps \mathcal{E}_D invariant, we can use Sweedler notation: $\widetilde{U}_J(a \otimes 1)\widetilde{U}_J^* = a_{(1)} \otimes a_{(2)}$, with $a, a_{(1)}$ belonging to \mathcal{E}_D , $a_{(2)}$ belonging to $\widetilde{\mathcal{Q}}_{\bar{J}}$, to have

$$\begin{aligned}
 & (\tau_R * h \otimes \text{id})\widetilde{U}_J(a \otimes \text{id})\widetilde{U}_J^* \\
 &= (\tau_R \otimes h)(\widetilde{U}_J(a_{(1)} \otimes \text{id})\widetilde{U}_J^*) \otimes a_{(2)} \\
 &= (\tau_R \otimes h \otimes \text{id})(\alpha_U \otimes \text{id})\alpha_U(a) \\
 &= (\tau_R \otimes h \otimes \text{id})(\text{id} \otimes \Delta)\alpha_U(a) \\
 &= (\tau_R \otimes \text{id})(\text{id} \otimes (h \otimes \text{id})\Delta)\alpha_U(a) \\
 &= (\tau_R \otimes \text{id})(\text{id} \otimes h(.).1)\alpha_U(a) \\
 &= (\text{id} \otimes h(.).1)(\tau_R \otimes \text{id})\alpha_U(a) \\
 &= (\text{id} \otimes h(.).1)\tau(a).1 \\
 &= \tau(a).1.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & (\tau_R \otimes \text{id})(\widetilde{U}_J(a \otimes 1)\widetilde{U}_J^*) \\
 &= (\tau_R * h \otimes \text{id})(\widetilde{U}_J(a \otimes 1)\widetilde{U}_J^*) = (\tau_R * h)(a_{(1)})a_{(2)} \\
 &= (\tau_R \otimes h \otimes \text{id})(a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)}) = (\tau_R \otimes h \otimes \text{id})(\text{id} \otimes \Delta_{\bar{J}}) \\
 & \quad (\widetilde{U}_J(a \otimes 1)\widetilde{U}_J^*) \\
 &= \tau_R(a_{(1)})(h \otimes \text{id}) \circ \Delta_{\bar{J}}(a_{(2)}) = \tau_R(a_{(1)})h(a_{(2)}).1_{\mathcal{Q}_{\bar{J}}} \\
 &= (\tau_R \otimes h)(a_{(1)} \otimes a_{(2)}) = (\tau_R * h)(a).1_{\mathcal{Q}_{\bar{J}}} = \tau_R(a).1_{\mathcal{Q}_{\bar{J}}}.
 \end{aligned}$$

□

Remark 4.3.6. If $QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D)$ ($QISO^+(\mathcal{A}^\infty, \mathcal{H}, D)$, if it exists) has a C^* action, then from the definition of a C^* action, we get a subalgebra \mathcal{A}_0 as in Lemma 4.3.1. Thus, the assumptions of section 4.2 are satisfied so that the conclusions in that subsection hold for $QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D)$ ($QISO^+(\mathcal{A}^\infty, \mathcal{H}, D)$). Similarly, the conclusions of Lemma 4.3.1 and the subsequent Lemmas hold for $QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D)$ ($QISO^+(\mathcal{A}^\infty, \mathcal{H}, D)$).

For any two compact quantum groups $(\mathcal{S}_1, U^{\mathcal{S}_1})$ and $(\mathcal{S}_2, U^{\mathcal{S}_2})$ in $\mathbf{Q}'(\mathcal{A}_J, \mathcal{H}, D)$, we write $\mathcal{S}_1 < \mathcal{S}_2$ if \mathcal{S}_1 is a sub object of \mathcal{S}_2 in the category $\mathbf{Q}'(\mathcal{A}_J, \mathcal{H}, D)$.

Lemma 4.3.7. *If G_1, G_2 be two CQG s such that $G_1 < G_2$ in the category $\mathbf{Q}'(\mathcal{A}_J, \mathcal{H}, D)$, If $(G_1)_{\tilde{J}}$ and $(G_2)_{\tilde{J}}$ make sense, then $(G_1)_J < (G_2)_J$ in the category $\mathbf{Q}'(\mathcal{A}_J, \mathcal{H}, D)$.*

Proof : From Corollary 4.3.4, we see that $(G_i)_{\tilde{J}}$ is an object in the category $\mathbf{Q}'(\mathcal{A}_J, \mathcal{H}, D)$. Let π_2 be the morphism from G_2 to G_1 in the category \mathcal{Q}' and π_1 be the morphism from G_1 to \mathbb{T}^n in the same category. Let $\Delta^i, \times_{\tilde{J}}^i, \chi^i$ denote respectively the coproducts, products and \mathbb{R}^{2n} action on $(G_i)_J, i = 1, 2$.

As the quantum group structure is not altered under Rieffel deformation, to prove the Lemma, it is enough to show that π_2 is a homomorphism from $(G_2)_J$ to $(G_1)_J$.

In any CQG (Q, Δ) , f, g linear functionals on Q and for all a in Q_0 , $(f \otimes \text{id})\Delta(\text{id} \otimes g)\Delta(a) = (f \otimes \text{id})\Delta(a_{(1)})g(a_{(2)}) = (f \otimes \text{id})(a_{(1)(1)} \otimes a_{(1)(2)})g(a_{(2)}) = f(a_{(1)(1)})g(a_{(2)})a_{(1)(2)} = (f \otimes \text{id} \otimes g)(a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_2) = (f \otimes \text{id} \otimes g)(\Delta(a_{(1)}) \otimes a_{(2)}) = (f \otimes \text{id} \otimes g)(\Delta \otimes \text{id})\Delta(a)$.

Hence,

$$(f \otimes \text{id})\Delta(\text{id} \otimes g)\Delta = (f \otimes \text{id} \otimes g)(\Delta \otimes \text{id})\Delta. \quad (4.3.1)$$

Moreover, we will also need the equation

$$(\pi_2 \otimes \pi_2)\Delta^2 = \Delta^1\pi_2, \quad (4.3.2)$$

which holds as π_2 is a morphism of CQG s, $G_2 \rightarrow G_1$.

Let λ, ρ be as in (1.3.6) and (1.3.7) and a belongs to $(G_2)_J$.

Then,

$$\begin{aligned} & \pi_2 \chi_{(s,u)}^2(a) \\ &= \pi_2(\lambda_{\eta(-s)}\rho_{\eta(u)})(a) \\ &= \pi_2(\text{ev}_{\eta(-s)}(\pi_1 \circ \pi_2) \otimes \text{id})\Delta^2(\text{id} \otimes \text{ev}_{\eta(u)}(\pi_1 \circ \pi_2))\Delta^2(a) \\ &= \pi_2(\text{ev}_{\eta(-s)}(\pi_1 \circ \pi_2) \otimes \text{id} \otimes \text{ev}_{\eta(u)} \circ (\pi_1 \circ \pi_2))(\Delta^2 \otimes \text{id})\Delta^2(a) \\ &\quad (\text{ by (4.3.1) }) \\ &= (\text{ev}_{\eta(-s)}(\pi_1 \circ \pi_2) \otimes \pi_2 \otimes \text{ev}_{\eta(u)} \circ (\pi_1 \circ \pi_2))(\Delta^2 \otimes \text{id})\Delta^2(a) \\ &= (\text{ev}_{\eta(-s)}\pi_1 \otimes \text{id} \otimes \text{ev}_{\eta(u)}\pi_1)(\pi_2 \otimes \pi_2 \otimes \pi_2)(\Delta^2 \otimes \text{id})\Delta^2(a) \\ &= (\text{ev}_{\eta(-s)}\pi_1 \otimes \text{id} \otimes \text{ev}_{\eta(u)}\pi_1)((\pi_2 \otimes \pi_2)\Delta^2 \otimes \pi_2)\Delta^2(a) \\ &= (\text{ev}_{\eta(-s)}\pi_1 \otimes \text{id} \otimes \text{ev}_{\eta(u)}\pi_1)(\Delta^1\pi_2 \otimes \pi_2)\Delta^2(a) \\ &\quad (\text{ using (4.3.2) }) \\ &= (\text{ev}_{\eta(-s)}\pi_1 \otimes \text{id} \otimes \text{ev}_{\eta(u)}\pi_1)(\Delta^1 \otimes \text{id})(\pi_2 \otimes \pi_2)\Delta^{(2)}(a) \\ &= (\text{ev}_{\eta(-s)}\pi_1 \otimes \text{id} \otimes \text{ev}_{\eta(u)}\pi_1)(\Delta^1 \otimes \text{id})\Delta^1(\pi_2(a)) \\ &= (\text{ev}_{\eta(-s)}\pi_1 \otimes \text{id})\Delta^1(\text{id} \otimes \text{ev}_{\eta(u)}\pi_1)\Delta^1(\pi_2(a)) \\ &= (\lambda_{\eta(-s)}\rho_{\eta(u)})\pi_2(a) \end{aligned}$$

$$= \chi_{(s,u)}^1 \pi_2(a).$$

Thus, for all (s, u) in \mathbb{R}^{2n} , $\pi_2 \chi_{(s,u)}^2 = \chi_{(s,u)}^1 \pi_2$.

Therefore, for all a, b in $(G_2)_J$,

$$\begin{aligned} \pi_2(a \times_J^2 b) &= \pi_2\left(\int \int \chi_{Ju}^2(a) \chi_v^2(b) e(u.v) dudv\right) \\ &= \int \int \pi_2(\chi_{Ju}^2(a)) \pi_2(\chi_v^2(b)) e(u.v) dudv \\ &= \int \int \chi_{Ju}^1(\pi_2(a)) \chi_v^1(\pi_2(b)) e(u.v) dudv \\ &= \pi_2(a) \times_J^1 \pi_2(b). \end{aligned}$$

where the third step is permissible by Proposition 1.3.2.

This proves that π_2 is indeed a homomorphism. \square

Theorem 4.3.8. 1. If $QISO_R^+(\mathcal{A}_J^\infty, \mathcal{H}, D)$ and $(QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D))_{\tilde{J}}$ have C^* actions on \mathcal{A} and \mathcal{A}_J respectively, we have

$$\widetilde{QISO_R^+(\mathcal{A}_J^\infty, \mathcal{H}, D)} \cong (\widetilde{QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D)})_{\tilde{J}}$$

$$QISO_R^+(\mathcal{A}_J^\infty, \mathcal{H}, D) \cong (QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D))_{\tilde{J}}.$$

2. If moreover, $\widetilde{QISO^+}(\mathcal{A}^\infty, \mathcal{H}, D)$ and $\widetilde{QISO^+}(\mathcal{A}_J^\infty, \mathcal{H}, D)$ both exist and have C^* actions on \mathcal{A} and \mathcal{A}_J respectively, then

$$\widetilde{QISO^+}(\mathcal{A}_J^\infty, \mathcal{H}, D) \cong \left(\widetilde{QISO^+}(\mathcal{A}^\infty, \mathcal{H}, D) \right)_{\tilde{J}},$$

$$QISO^+(\mathcal{A}_J^\infty, \mathcal{H}, D) \cong (QISO^+(\mathcal{A}^\infty, \mathcal{H}, D))_{\tilde{J}}.$$

Proof : We prove 1 only. From Corollary 4.3.4 and Lemma 4.3.5 we see that $\widetilde{QISO_R^+}(\mathcal{A}, \mathcal{H}, D)_{\tilde{J}}$ is an object of $\mathbf{Q}'_R(\mathcal{A}_J, \mathcal{H}, D)$. Thus,

$$(\widetilde{QISO_R^+}(\mathcal{A}, \mathcal{H}, D))_{\tilde{J}} < \widetilde{QISO_R^+}(\mathcal{A}_J, \mathcal{H}, D) \text{ in } \mathbf{Q}'_R(\mathcal{A}_J, \mathcal{H}, D).$$

So, by Lemma 4.3.7, $((\widetilde{QISO_R^+}(\mathcal{A}, \mathcal{H}, D))_{\tilde{J}})_{-\tilde{J}} < (\widetilde{QISO_R^+}(\mathcal{A}_J, \mathcal{H}, D))_{-\tilde{J}}$ in $\mathbf{Q}'_R(\mathcal{A}, \mathcal{H}, D)$, hence $\widetilde{QISO_R^+}(\mathcal{A}, \mathcal{H}, D) < (\widetilde{QISO_R^+}(\mathcal{A}_J, \mathcal{H}, D))_{-\tilde{J}}$.

Replacing \mathcal{A} by \mathcal{A}_{-J} , we have

$$\begin{aligned} & \widetilde{QISO}_R^+(\mathcal{A}_{-J}, \mathcal{H}, D) \\ & < \widetilde{QISO}_R^+((\mathcal{A}_{-J})_J, \mathcal{H}, D)_{-\tilde{J}} \text{ (in } \mathbf{Q}'_R(\mathcal{A}_{-J}, \mathcal{H}, D)) \\ & \cong \widetilde{QISO}_R^+(\mathcal{A}, \mathcal{H}, D)_{-\tilde{J}} \text{ (in } \mathbf{Q}'_R(\mathcal{A}_{-J}, \mathcal{H}, D)) \cong (\widetilde{QISO}_R^+(\mathcal{A}, \mathcal{H}, D))_{-\tilde{J}}. \end{aligned}$$

Thus, $\widetilde{QISO}_R^+(\mathcal{A}_J, \mathcal{H}, D) < (\widetilde{QISO}_R^+(\mathcal{A}, \mathcal{H}, D))_{\tilde{J}}$ in $\mathbf{Q}'_R(\mathcal{A}_J, \mathcal{H}, D)$ which implies $\widetilde{QISO}_R^+(\mathcal{A}_J, \mathcal{H}, D) \cong (\widetilde{QISO}_R^+(\mathcal{A}, \mathcal{H}, D))_{\tilde{J}}$ in $\mathbf{Q}'_R(\mathcal{A}_J, \mathcal{H}, D)$. \square

4.3.2 Computations

Fix a real number θ , and then we recall from subsection 1.1.1 that the C^* algebra \mathcal{A}_θ is the universal C^* algebra generated by two unitaries U and V such that $UV = \lambda VU$, where $\lambda := e^{2\pi i\theta}$. We also recall from section 1.3 that \mathcal{A}_θ is a Rieffel type deformation of $C(\mathbb{T}^2)$ by using the canonical action of \mathbb{R}^2 on \mathbb{T}^2 and the skew symmetric matrix $J = \begin{pmatrix} 0 & -\frac{\theta}{2} \\ \frac{\theta}{2} & 0 \end{pmatrix}$. It is well-known (see [17]) that the set $\{U^m V^n : m, n \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(\mathcal{A}_\theta, \tau)$, where τ denotes the unique faithful normalized trace on \mathcal{A}_θ given by, $\tau(\sum a_{mn} U^m V^n) = a_{00}$. We will denote the GNS space $L^2(\mathcal{A}_\theta, \tau)$ by \mathcal{H}_0 . Let $\mathcal{A}_\theta^{\text{fin}}$ be the unital $*$ -subalgebra generated by finite complex linear combinations of $U^m V^n$, where m, n belong to \mathbf{Z} , and d_1, d_2 be the maps on $\mathcal{A}_\theta^{\text{fin}}$ defined by $d_1(U^m V^n) = m U^m V^n$, $d_2(U^m V^n) = n U^m V^n$.

We consider the spectral triple obtained from the classical spectral triple on \mathbb{T}^2 as described in section 4.1.

Theorem 4.3.9. $\widetilde{QISO}^+(\mathcal{A}_\theta^\infty, \mathcal{H}, D) = \widetilde{QISO}^+(C^\infty(\mathbb{T}^2)) = C(\mathbb{T}^2) * C(\mathbb{T})$, and $QISO^+(\mathcal{A}_\theta^\infty) = QISO^+(C^\infty(\mathbb{T}^2)) = C(\mathbb{T}^2)$.

Proof: We use Theorem 4.3.8 and recall that $QISO^+(C^\infty(\mathbb{T}^2)) = C(\mathbb{T}^2)$ (Theorem 3.4.14) which is generated by z_1 and z_2 , say. $QISO^+(C^\infty(\mathbb{T}^2))$ contains $C(\mathbb{T}^2)$ itself as a quantum subgroup. Hence, by Theorem 4.3.8, $QISO^+(\mathcal{A}_\theta^\infty)$ is the CQG obtained from the Rieffel deformation of $C(\mathbb{T}^2)$ via the action of \mathbb{R}^4 and the skew symmetric matrix

$$\tilde{J} = J \oplus -J = \begin{pmatrix} 0 & -\frac{\theta}{2} & 0 & 0 \\ \frac{\theta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\theta}{2} \\ 0 & 0 & -\frac{\theta}{2} & 0 \end{pmatrix} \text{ so that } \tilde{J}(r_1, r_2, r_3, r_4) = (-\frac{\theta}{2}r_2, \frac{\theta}{2}r_1, \frac{\theta}{2}r_4, -\frac{\theta}{2}r_3).$$

For f_1, f_2 in $C^\infty(\mathbb{T}^2)$, $r = (r_1, r_2, r_3, r_4)$ in \mathbb{R}^4 , $r' = (r'_1, r'_2, r'_3, r'_4)$ in \mathbb{R}^4 , (t_1, t_2) in \mathbb{T}^2 ,

the deformed product is given by

$$(f_1 \times_{\tilde{J}} f_2)(t_1, t_2) = \int \int \chi_{\tilde{J}(r_1, r_2, r_3, r_4)}(f_1)(t_1, t_2) \chi_{(r'_1, r'_2, r'_3, r'_4)}(f_2)(t_1, t_2) e(r.r') dr dr'.$$

Here, for f in $C^\infty(\mathbb{T}^2)$,

$$\begin{aligned} & \chi_{(r_1, r_2, r_3, r_4)} f(t_1, t_2) \\ &= (ev_{\eta(-(r_1, r_2))} \otimes \text{id}) \Delta(\text{id} \otimes ev_{\eta(r_3, r_4)}) \Delta(f)(t_1, t_2) \\ &= f(\eta(-(r_1, r_2))(t_1, t_2) \eta(r_3, r_4)) \\ &= f((e(-r_1), e(-r_2))(t_1, t_2) (e(r_3), e(r_4))) \\ &= f(e(r_3 - r_1)t_1, e(r_4 - r_2)t_2). \end{aligned}$$

Therefore,

$$\begin{aligned} & z_1 \times_{\tilde{J}} z_2 \\ &= \int \int z_1(e(\frac{\theta}{2}r_4 + \frac{\theta}{2}r_2)t_1, e(-\frac{\theta}{2}r_1 - \frac{\theta}{2}r_3)t_2) z_2(e(r'_3 - r'_1)t_1, e(-r'_2 + r'_4)t_2) \\ & \quad e(r.r') dr dr' \\ &= \int \int e(\frac{\theta}{2}r_4 + \frac{\theta}{2}r_2)t_1 e(-r'_2 + r'_4)t_2 e(r_1.r'_1) e(r_2.r'_2) \cdot e(r_3.r'_3) e(r_4.r'_4) dr dr' \\ &= t_1 t_2 \int \int e(\frac{\theta}{2}r_2) e(-r'_2) e(r_2.r'_2) dr_2 dr'_2 \int \int e(\frac{\theta}{2}r_4) e(r'_4) e(r_4.r'_4) dr_4 dr'_4 \\ & \quad \int \int e(r_1.r'_1) dr_1 dr'_1 \int \int e(r_3.r'_3) dr_3 dr'_3 \\ &= t_1 t_2 \int \int e(-\frac{\theta}{2}r_2) e(r'_2) e((-r_2).(-r'_2)) dr_2 dr'_2 \int \int e(-\frac{\theta}{2}r_4) e(-r'_4) e(-r_4. - r'_4) \\ & \quad dr_4 dr'_4. 1.1. \end{aligned}$$

(by Proposition 1.3.1)

Similarly,

$$\begin{aligned} & z_2 \times_{\tilde{J}} z_1 \\ &= \int \int e(-\frac{\theta}{2}r_1 - \frac{\theta}{2}r_3)t_2 e(r'_3 - r'_1)t_1 e(r_1.r'_1) e(r_3.r'_3) dr_1 dr'_1 dr_3 dr'_3. 1.1 \\ &= t_1 t_2 \int \int e(-\frac{\theta}{2}r_1) e(-r'_1) e(r_1.r'_1) dr_1 dr'_1 \int \int e(-\frac{\theta}{2}r_3) e(r'_3) e(r_3.r'_3) dr_3 dr'_3 \\ &= z_1 \times_{\tilde{J}} z_2. \end{aligned}$$

This proves the theorem. \square

4.4 QISO^ℒ of a Rieffel deformed noncommutative manifold

4.4.1 Derivation of the main result

In this subsection, our set up is as in section 4.1 so that we have spectral triples on \mathcal{A}_J^∞ for each J such that the spectral triple on \mathcal{A}^∞ satisfies all the assumptions mentioned in chapter 2 for ensuring the existence of QISO^ℒ where \mathcal{L} is the Laplacian coming from the spectral triple. As QISO^ℒ has a C^* action on \mathcal{A} , we get a subalgebra \mathcal{A}_0 as in Lemma 4.3.1. Thus, the assumptions of section 4.2 are satisfied so that the conclusions in that subsection hold for QISO^ℒ($\mathcal{A}^\infty, \mathcal{H}, D$).

We recall from Chapter 2 that \mathcal{A}_0 will denote the $*$ -algebra generated by complex linear (algebraic, not closed) span \mathcal{A}_0^∞ of the eigenvectors of \mathcal{L} which has a countable discrete set of eigenvalues each with finite multiplicities, by assumptions for the existence of QISO^ℒ. By the same assumptions, \mathcal{A}_0^∞ is a subset of \mathcal{A}^∞ and is norm-dense in \mathcal{A} . Here, we make the following additional assumptions.

Assumptions (i) \mathcal{A}_0 is dense in \mathcal{A}^∞ w.r.t. the Frechet topology coming from the action of \mathbb{T}^n .

(ii) $\bigcap_{n \geq 1} \text{Dom}(\mathcal{L}^n) = \mathcal{A}^\infty$.

(iii) \mathcal{L} commutes with the \mathbb{T}^n -action β , hence $C(\mathbb{T}^n)$ can be identified as a sub object of QISO^ℒ in the category $\mathbf{Q}'_{\mathcal{L}}$.

Let π denote the surjective map from QISO^ℒ to its quantum subgroup $C(\mathbb{T}^n)$, which is a morphism of compact quantum groups. We denote by $\alpha : \mathcal{A} \rightarrow \mathcal{A} \otimes \text{QISO}^\mathcal{L}$ the action of QISO^ℒ on \mathcal{A} , and note that on \mathcal{A}_0 , this action is algebraic, that is, it is an action of the Hopf $*$ -algebra \mathcal{Q}_0 consisting of matrix elements of finite dimensional unitary representations of \mathcal{Q} . We have $(\text{id} \otimes \pi) \circ \alpha = \beta$.

We recall from Proposition 1.3.8 that $\mathcal{A}^\infty = \mathcal{A}_J^\infty$ as topological spaces, that is they coincide as sets and the corresponding Frechet topologies are also equivalent. In view of this, we shall denote this space simply by \mathcal{A}^∞ , unless one needs to consider it as Frechet algebra, in which case the suffix J will be used.

Lemma 4.4.1. For F in $\mathcal{B}^{\mathcal{B}(\mathcal{H})}(\mathbb{R}^2)$ (notation as in section 1.3) and a trace class operator W ,

$$\text{Tr} \left(\int \int F(u, v) W e(u.v) du dv \right) = \int \int \text{Tr}(F(u, v) W) e(u.v) du dv.$$

Proof : From the definition of oscillatory integral, we have $\pi_J(F(u, v)) = \sum_{p \in L} \int (F(u, v) \phi_p)(u, v) e(u.v) du dv$ (notations as in section 1.3). Let $\{L_n\}_n$ be a sequence of subsets of L such that it increases to L . Then, as the above sum is strongly

convergent, $\lim_n(\sum_{p \in L_n} \int (F(u, v) \phi_p)(u, v) e(u, v) du dv)$ converges in SOT. We deduce the result by using Lemma 1.1.5. \square

Proposition 4.4.2. *Let \mathcal{L}_J denote the Laplacian from the spectral triple $(\mathcal{A}_J^\infty, \mathcal{H}, D)$. Then \mathcal{L}_J coincides with \mathcal{L} on $\mathcal{A}^\infty \subseteq \mathcal{A}_J$.*

Proof : We recall that from the proof of Lemma 4.1.4, we have $[D, \pi_J(a)] = \pi_J([D, a])$. Denoting the inner product on $\Omega_J^1(\mathcal{A}^\infty)$ by $\langle \cdot, \cdot \rangle_J$ and letting a, b in \mathcal{A}^∞ , Lim as in subsection 1.5.2, we have by using Lemma 4.4.1

$$\begin{aligned}
& \langle \mathcal{L}_J a, b \rangle_J \\
&= \lim_{t \rightarrow 0^+} \frac{\text{Tr}([D, \pi_J(a)]^* [D, \pi_J(b)] e^{-tD^2})}{\text{Tr}(e^{-tD^2})} \\
&= \lim_{t \rightarrow 0^+} \frac{\text{Tr}(\pi_J([D, a]^* [D, b]) e^{-tD^2})}{\text{Tr}(e^{-tD^2})} \\
&= \lim_{t \rightarrow 0^+} \frac{\text{Tr}(\int \beta_{Ju}([D, a]^* [D, b]) \widetilde{\beta}_v e(u, v) du dv e^{-tD^2})}{\text{Tr}(e^{-tD^2})} \\
&= \lim_{t \rightarrow 0^+} \frac{\int \int \text{Tr}(\beta_{Ju}([D, a]^* [D, b]) \widetilde{\beta}_v e^{-tD^2}) e(u, v) du dv}{\text{Tr}(e^{-tD^2})} \\
&\quad (\text{ by Lemma 4.4.1}) \\
&= \lim_{t \rightarrow 0^+} \frac{\int \int \text{Tr}(V_{\widetilde{Ju}}([D, a]^* [D, b]) V_{\widetilde{Ju}}^{-1} \beta_v e^{-tD^2}) e(u, v) du dv}{\text{Tr}(e^{-tD^2})} \\
&= \lim_{t \rightarrow 0^+} \frac{\int \int \text{Tr}(V_{\widetilde{Ju}}([D, a]^* [D, b]) \beta_v e^{-tD^2} V_{\widetilde{Ju}}^{-1}) e(u, v) du dv}{\text{Tr}(e^{-tD^2})} \\
&= \lim_{t \rightarrow 0^+} \frac{\int \int \text{Tr}([D, a]^* [D, b]) \widetilde{\beta}_v e^{-tD^2} e(u, v) du dv}{\text{Tr}(e^{-tD^2})} \\
&= \lim_{t \rightarrow 0^+} \frac{\text{Tr}(\int \int [D, a]^* [D, b] \widetilde{\beta}_v e^{-tD^2} e(u, v) du dv)}{\text{Tr}(e^{-tD^2})} \\
&\quad \text{which by Proposition 1.3.1, equals} \\
&= \lim_{t \rightarrow 0^+} \frac{\text{Tr}([D, a]^* [D, b] e^{-tD^2})}{\text{Tr}(e^{-tD^2})} \\
&= \langle \mathcal{L} a, b \rangle.
\end{aligned}$$

\square

Thus, the quantum isometry group $QISO^\mathcal{L}(\mathcal{A}_J)$ is the universal compact quantum group acting on \mathcal{A}_J , with the action keeping each of the eigenspaces of \mathcal{L} invariant. Note that the algebraic span of eigenvectors of \mathcal{L}_J coincides with that of \mathcal{L} , that is \mathcal{A}_0^∞ , which is already assumed to be Frechet-dense in $\mathcal{A}^\infty = \mathcal{A}_J^\infty$, hence in particular norm-dense in \mathcal{A}_J . Moreover, all the results of section 4.2 hold for $QISO^\mathcal{L}(\mathcal{A}_J)$.

We now state and prove a criterion, to be used later, for extending positive maps defined on \mathcal{A}_0 .

Lemma 4.4.3. *Let \mathcal{B} be another unital C^* -algebra equipped with a \mathbb{T}^n -action, so that we can consider the C^* -algebras \mathcal{B}_J for any skew symmetric $n \times n$ matrix J . Let $\phi : \mathcal{A}^\infty \rightarrow \mathcal{B}^\infty$ be a linear map, satisfying the following :*

(a) *ϕ is positive w.r.t. the deformed products \times_J on \mathcal{A}_0 and \mathcal{B}^∞ , that is $\phi(a^* \times_J a) \geq 0$ (in $\mathcal{B}_J^\infty \subset \mathcal{B}_J$) for all a in \mathcal{A}_0 , and*

(b) *ϕ extends to a norm-bounded map (say ϕ_0) from \mathcal{A} to \mathcal{B} .*

Then ϕ also have an extension ϕ_J as a $\| \cdot \|_J$ -bounded positive map from \mathcal{A}_J to \mathcal{B}_J satisfying $\|\phi_J\| = \|\phi(1)\|_J$.

Proof : We can view ϕ as a map between the Frechet spaces \mathcal{A}^∞ and \mathcal{B}^∞ , which is clearly closable, since it is continuous w.r.t. the norm-topologies of \mathcal{A} and \mathcal{B} , which are weaker than the corresponding Frechet topologies. By the Closed Graph Theorem, we conclude that ϕ is continuous in the Frechet topology. Since $\mathcal{A}^\infty = \mathcal{A}_J^\infty$ and $\mathcal{B}^\infty = \mathcal{B}_J^\infty$ as Frechet spaces, consider ϕ as a continuous map from \mathcal{A}_J^∞ to \mathcal{B}_J^∞ , and it follows by the Frechet-continuity of \times_J and $*$ and the Frechet-density of \mathcal{A}_0 in \mathcal{A}_J^∞ that the positivity (w.r.t. \times_J) of the restriction of ϕ to $\mathcal{A}_0 \subset \mathcal{A}_J^\infty$ is inherited by the extension on $\mathcal{A}^\infty = \mathcal{A}_J^\infty$. Indeed, given a in $\mathcal{A}_J^\infty = \mathcal{A}^\infty$, choose a sequence a_n belonging to \mathcal{A}_0 such that $a_n \rightarrow a$ in the Frechet topology. We have $\phi(a^* \times_J a) = \lim_n \phi(a_n^* \times_J a_n)$ in the Frechet topology, so in particular, $\phi(a_n^* \times_J a_n) \rightarrow \phi(a^* \times_J a)$ in the norm of \mathcal{B}_J , which implies that $\phi(a^* \times_J a)$ is a positive element of \mathcal{B}_J since $\phi(a_n^* \times_J a_n)$ is so for each n . Next, by Lemma 4.1.2, we note that \mathcal{A}^∞ is closed under holomorphic functional calculus as a unital $*$ -subalgebra of \mathcal{A}_J (the identity of \mathcal{A}_J^∞ is same as that of \mathcal{A}), and hence, by Lemma 4.1.2, for any self adjoint element x in \mathcal{A}^∞ , $\|\phi(x)\| \leq \|x\| \phi(1)$. Thus, for any y in \mathcal{A}^∞ , $\|\phi(y)\| = \left\| \phi\left(\frac{y+y^*}{2} + i\frac{y-y^*}{2i}\right) \right\| \leq \left\| \phi\left(\frac{y+y^*}{2}\right) \right\| + \left\| \phi\left(\frac{y-y^*}{2i}\right) \right\| \leq \left(\left\| \frac{y+y^*}{2} \right\| + \left\| \frac{y-y^*}{2i} \right\| \right) \phi(1) \leq 2\|y\| \phi(1)$. Thus, ϕ is bounded on \mathcal{A}^∞ and hence admits a bounded extension (say ϕ_J) on \mathcal{A}_J , which will still be a positive map, so in particular the norm of ϕ_J is same as $\|\phi_J(1)\|$. \square

Now we note that due to the assumption (iii), $(QISO^L)_{\tilde{J}}$ makes sense.

Remark 4.4.4. *By Lemma 4.2.7 along with Lemma 4.2.8, α is a homomorphism from \mathcal{A}_J to $\mathcal{A}_J \otimes (QISO^L(\mathcal{A}))_{\tilde{J}}$ and hence $(QISO^L(\mathcal{A}))_{\tilde{J}}$ is an object in the category $\mathcal{Q}^L(\mathcal{A}_J)$.*

Theorem 4.4.5. *If the Haar state is faithful on \mathcal{Q} , then $\alpha : \mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{Q}_0$ extends to an action of the compact quantum group $\mathcal{Q}_{\tilde{J}}$ on \mathcal{A}_J , which is isometric, smooth and faithful.*

Proof : We have already seen in (4.2.10) that α is an algebra homomorphism from \mathcal{A}_0 to $\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{Q}_0$ (w.r.t. the deformed products), and it is also $*$ -homomorphism since it is so for the undeformed case and the involution $*$ is the same for the deformed and undeformed algebras. It now suffices to show that α extends to \mathcal{A}_J as a C^* -homomorphism. Let us fix any faithful imbedding $\mathcal{A}_J \subseteq \mathcal{B}(\mathcal{H}_0)$ (where \mathcal{H}_0 is a Hilbert space) and consider the imbedding $\mathcal{Q}_{\tilde{J}} \subseteq \mathcal{B}(L^2(h_J))$ (by virtue of Lemma 1.3.13). By definition, the norm on $\mathcal{A}_J \otimes \mathcal{Q}_{\tilde{J}}$ is the minimal (injective) C^* -norm, so it is equal to the norm inherited from the imbedding $\mathcal{A}_J \otimes_{\text{alg}} \mathcal{Q}_{\tilde{J}} \subseteq \mathcal{B}(\mathcal{H}_0 \otimes L^2(h_J))$. Let us consider the dense subspace $\mathcal{D} \subset \mathcal{H}_0 \otimes L^2(h_J)$ consisting of vectors which are finite linear combinations of the form $\sum_i u_i \otimes x_i$, with u_i belonging to \mathcal{H}_0 , x_i belonging to $\mathcal{Q}_0 \subset L^2(h_J)$. Fix such a vector $\xi = \sum_{i=1}^k u_i \otimes x_i$ and consider $\mathcal{B} := \mathcal{A} \otimes M_k(\mathbb{C})$, with the \mathbb{T}^n -action $\beta \otimes \text{id}$ on \mathcal{B} . Let $\phi : \mathcal{A}^\infty \rightarrow \mathcal{B}^\infty$ be the map given by

$$\phi(a) := \left(\left((\text{id} \otimes \phi_{(x_i, x_j)})(\alpha(a)) \right) \right)_{i,j=1}^k,$$

where $\phi_{(x,y)}(z) := h(x^* \times_{\tilde{J}} z \times_{\tilde{J}} y)$ for x, y, z in \mathcal{Q}_0 . By Remark 1.3.11, $\phi_{(x_i, x_j)}$ extends to \mathcal{Q} as a bounded linear functional. Note that the range of ϕ is in $\mathcal{B}^\infty = \mathcal{A}^\infty \otimes M_k(\mathbb{C})$ since we have $(\text{id} \otimes \phi_{(x,y)})\alpha(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$ by Proposition 2.1.2, using our assumption (ii) that $\bigcap_{n \geq 1} \text{Dom}(\mathcal{L}^n) = \mathcal{A}^\infty$.

Since α maps \mathcal{A}_0 into $\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{Q}_0$ and $h = h_J$ on \mathcal{Q}_0 , it is easy to see that for a in \mathcal{A}_0 , $\phi(a^* \times_J a)$ is positive in \mathcal{B}_J . As $\phi_{(x_i, x_j)}$ extends to \mathcal{Q} as a bounded linear functional, ϕ extends to a bounded linear (but not necessarily positive) map from \mathcal{A} to \mathcal{B} . Thus, the hypotheses of Lemma 4.4.3 are satisfied and we conclude that ϕ admits a positive extension, say ϕ_J , from \mathcal{A}_J to $\mathcal{B}_J = \mathcal{A}_J \otimes M_k(\mathbb{C})$. Thus, we have for a in \mathcal{A}_0 ,

$$\begin{aligned} & \sum_{i,j=1}^k \langle u_i, \phi(a^* \times_J a) u_j \rangle \\ & \leq \|a\|_J^2 \sum_{ij} \langle u_i, \phi(1) u_j \rangle = \|a\|_J^2 \sum_{ij} \langle u_i, u_j \rangle h(x_i^* \times_{\tilde{J}} x_j) \\ & = \|a\|_J^2 \sum_{ij} \langle u_i \otimes x_i, u_j \otimes x_j \rangle = \|a\|_J^2 \left\| \sum_{i=1}^k u_i \otimes x_i \right\|^2. \end{aligned}$$

This implies

$$\|\alpha(a)\xi\|^2 = \langle \xi, \alpha(a^* \times_J a)\xi \rangle = \sum_{i,j=1}^k \langle u_i, \phi(a^* \times_J a) u_j \rangle \leq \|a\|_J^2 \|\xi\|^2$$

for all ξ in \mathcal{D} and a in \mathcal{A}_0 , hence α admits a bounded extension which is clearly a C^* -homomorphism. \square

For any two objects \mathcal{S}_1 and \mathcal{S}_2 in $\mathbf{Q}'_{\mathcal{L}}$, we write $\mathcal{S}_1 <_{\mathcal{L}} \mathcal{S}_2$ if \mathcal{S}_1 is a sub-object of \mathcal{S}_2 in the category $\mathbf{Q}'_{\mathcal{L}}$.

Lemma 4.4.6. *If G_1, G_2 be two CQG's such that $G_1 <_{\mathcal{L}} G_2$ in the category $\mathbf{Q}'_{\mathcal{L}}(\mathcal{A}, \mathcal{H}, D)$. If $(G_1)_{\tilde{J}}$ and $(G_2)_{\tilde{J}}$ make sense, then $(G_1)_{\tilde{J}} <_{\mathcal{L}} (G_2)_{\tilde{J}}$ in the category $\mathbf{Q}'_{\mathcal{L}}(\mathcal{A}_J, \mathcal{H}, D)$.*

Proof : By Remark 4.4.4, $(G_i)_{\tilde{J}}$ are objects in the category $\mathbf{Q}'_{\mathcal{L}}(\mathcal{A}_J)$, $i = 1, 2$. The rest of the proof is the same as that of Lemma 4.3.7 and hence we omit it. \square

Theorem 4.4.7. *If the Haar state on $QISO^{\mathcal{L}}(\mathcal{A})$ is faithful, we have the isomorphism of compact quantum groups:*

$$(QISO^{\mathcal{L}}(\mathcal{A}))_{\tilde{J}} \cong QISO^{\mathcal{L}}(\mathcal{A}_J).$$

Proof : By Theorem 4.4.5, we have seen that $(QISO^{\mathcal{L}}(\mathcal{A}))_{\tilde{J}}$ also acts faithfully, smoothly and isometrically on \mathcal{A}_J , which implies,

$$(QISO^{\mathcal{L}}(\mathcal{A}, \mathcal{H}, D))_{\tilde{J}} < QISO^{\mathcal{L}}(\mathcal{A}_J, \mathcal{H}, D) \text{ in } \mathbf{Q}'_{\mathcal{L}}(\mathcal{A}_J, \mathcal{H}, D).$$

The rest of the proof is same as that of Theorem 4.3.8 by using Lemma 4.4.6 and hence we omit it. \square

4.4.2 Computations

The case of the noncommutative tori

We recall (chapter 1) that the noncommutative n -tori \mathbb{T}_{θ}^n is the universal C^* algebra generated by n unitaries U_1, U_2, \dots, U_n such that $U_i U_j = e(\theta_{ij}) U_j U_i$, $i, j = 1, 2, \dots, n$ where $\theta \equiv ((\theta_{ij}))$ is a $n \times n$ skew symmetric matrix. We also recall that \mathbb{T}_{θ}^n is a Rieffel deformation of $C(\mathbb{T}^n)$ via the \mathbb{R}^n action on $C(\mathbb{T}^n)$ (section 1.3) and the matrix $J = \frac{\theta}{2}$.

We consider the isospectral deformation of the classical spectral triple on $C(\mathbb{T}^n)$ which will give a spectral triple on \mathbb{T}_{θ}^n . so that the corresponding Laplacian \mathcal{L} is given by $\mathcal{L}(U_1^{m_1} \dots U_n^{m_n}) = -(m_1^2 + \dots m_n^2) U_1^{m_1} \dots U_n^{m_n}$, and it is also easy to see that all the assumptions in chapter 2 required for defining $QISO^{\mathcal{L}}(\mathbb{T}_{\theta}^n)$ are satisfied.

Theorem 4.4.8. *Using Theorem 4.4.7, we conclude that $QISO^{\mathcal{L}}(\mathbb{T}_{\theta}^n)$ is a Rieffel deformation of $QISO^{\mathcal{L}}(C(\mathbb{T}^n))$.*

Next, we will use Theorem 4.4.7 to compute the exact structure of $QISO^{\mathcal{L}}(\mathcal{A}_{\theta})$. Our target is to show that $QISO^{\mathcal{L}}(\mathcal{A}_{\frac{1}{2}})$ is a commutative C^* algebra which will give an example of a non commutative C^* algebra with $QISO^{\mathcal{L}}$ a commutative CQG.

We have seen (Theorem 2.2.17) that $QISO^{\mathcal{L}}(C(\mathbb{T}^n)) \cong C(\mathbb{T}^n \rtimes (\mathbf{Z}_2^n \rtimes S_n))$.

In particular, $QISO^{\mathcal{L}}(C(\mathbb{T}^2)) \cong C(\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes S_2))$ where the definition of the two semi-direct products are described below. Let the generators of the first, second and the third copy of \mathbf{Z}_2 in $\mathbf{Z}_2^2 \rtimes S_2$ be denoted by $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Therefore, as a set $(\mathbf{Z}_2^2 \rtimes S_2)$ equals $\{(\gamma_1, \gamma_2, \gamma_3) : \gamma_i \in \{0, 1\}, i = 1, 2, 3\}$ and $\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes S_2) = \{(z_1, z_2, \gamma_1, \gamma_2, \gamma_3) : z_1, z_2 \in S^1, \gamma_i \in \{0, 1\}, i = 1, 2, 3\}$. The action of \mathbf{Z}_2 on \mathbf{Z}_2^2 is given by $(0, 0, 1)(x, y) = (y, x)$ ((x, y) belongs to \mathbb{T}^2) and the action of $\mathbf{Z}_2^2 \rtimes S_2$ on \mathbb{T}^2 (denoted by \circ) is given by

$$(1, 0, 0) \circ (z_1, z_2) = (\overline{z_1}, z_2), (0, 1, 0) \circ (z_1, z_2) = (z_1, \overline{z_2}), (0, 0, 1) \circ (z_1, z_2) = (z_2, z_1).$$

Lemma 4.4.9. *Let (z_1, z_2) , (z'_1, z'_2) , (z''_1, z''_2) belong to \mathbb{T}^2 . Then we have the following group multiplication formulae in $\mathbb{T}^2 \rtimes (\mathbf{Z}_2^2 \rtimes S_2)$:*

$$(z_1, z_2, 0, 0, 0)(z'_1, z'_2, 1, 0, 1)(z''_1, z''_2, 0, 0, 0) = (z_1 z'_1 \overline{z''_2}, z_2 z'_2 z''_1, 1, 0, 1), \quad (4.4.1)$$

$$(z_1, z_2, 0, 0, 0)(z'_1, z'_2, 0, 0, 1)(z''_1, z''_2, 0, 0, 0) = (z_1 z'_1 z''_2, z_2 z'_2 z''_1, 0, 0, 1), \quad (4.4.2)$$

$$(z_1, z_2, 0, 0, 0)(z'_1, z'_2, 0, 0, 0)(z''_1, z''_2, 0, 0, 0) = (z_1 z'_1 z''_1, z_2 z'_2 z''_2, 0, 0, 0). \quad (4.4.3)$$

Proof : Using the definition of semi direct product , we have

$$\begin{aligned} & (z_1, z_2, 0, 0, 0)(z'_1, z'_2, 1, 0, 1)(z''_1, z''_2, 0, 0, 0) \\ &= (z_1 z'_1, z_2 z'_2, 1, 0, 1)(z''_1, z''_2, 0, 0, 0) \\ &= (z_1 z'_1, z_2 z'_2)((1, 0, 1) \circ (z''_1, z''_2), 1, 0, 1) \\ &= (z_1 z'_1, z_2 z'_2)((1, 0), 0)((0, 0), 1) \circ (z''_1, z''_2), 1, 0, 1) \\ &= ((z_1 z'_1, z_2 z'_2)(\overline{z''_2}, z''_1), 1, 0, 1) \\ &= (z_1 z'_1 \overline{z''_2}, z_2 z'_2 z''_1, 1, 0, 1). \end{aligned}$$

Thus, we have (4.4.1).

Similarly,

$$\begin{aligned} & (z_1, z_2, 0, 0, 0)(z'_1, z'_2, 0, 0, 1)(z''_1, z''_2, 0, 0, 0) \\ &= (z_1 z'_1, z_2 z'_2, 0, 0, 1)(z''_1, z''_2, 0, 0, 0) \\ &= ((z_1 z'_1, z_2 z'_2)((0, 0, 1) \circ (z''_1, z''_2)), 0, 0, 1) \\ &= ((z_1 z'_1, z_2 z'_2)(z''_2, z''_1), 0, 0, 1) \\ &= (z_1 z'_1 z''_2, z_2 z'_2 z''_1, 0, 0, 1), \end{aligned}$$

which proves (4.4.2).

(4.4.3) follows similarly and therefore we omit the proof. \square

$QISO^L(C(\mathbb{T}^2)) = C(\mathbb{T}^2 \succ \triangleleft (\mathbf{Z}_2^2 \succ \triangleleft \mathbf{Z}_2))$ which as a C^* algebra is isomorphic to $\oplus_{i=1,2,\dots,8} C(\mathbb{T}^2)$. We will denote the generators of $QISO^L(\mathbb{T}^2)$ by $\{A_{(\gamma_1, \gamma_2, \gamma_3)}, B_{(\gamma_1, \gamma_2, \gamma_3)} : (\gamma_1, \gamma_2, \gamma_3) \in \mathbf{Z}_2^3\}$, more precisely, we fix the following convention:

$$A_{(\gamma_1, \gamma_2, \gamma_3)}(z_1, z_2, \gamma_1, \gamma_2, \gamma_3) = z_1, \quad B_{(\gamma_1, \gamma_2, \gamma_3)}(z_1, z_2, \gamma_1, \gamma_2, \gamma_3) = z_2.$$

Now, \mathbb{T}^2 sits as a subgroup of $\mathbb{T}^2 \succ \triangleleft \mathbf{Z}_2^3$ as $\{(z_1, z_2, 0, 0, 0) : z_i \in S^1\}$.

Hence, the action of \mathbb{T}^2 on $\mathbb{T}^2 \succ \triangleleft \mathbf{Z}_2^3$ with which we are concerned with is given by the group multiplication in $\mathbb{T}^2 \succ \triangleleft \mathbf{Z}_2^3$, that is, the action of (z_1, z_2) ($(z_i \in S^1)$) on $(z'_1, z'_2, \gamma_1, \gamma_2, \gamma_3) \in \mathbb{T}^2 \succ \triangleleft \mathbf{Z}_2^3$ is given by $(z_1, z_2, 0, 0, 0)(z'_1, z'_2, \gamma_1, \gamma_2, \gamma_3)$.

The action of \mathbb{R}^4 on $QISO^L(C(\mathbb{T}^2))$ as prescribed by Wang (1.3.8) is given by

$$\alpha_{(s,u)} = (\Omega(-s) \otimes \text{id}) \Delta(\text{id} \otimes \Omega(u)) \Delta$$

where s, u belong to \mathbb{R}^2 .

If $s = (s_1, s_2)$, $u = (u_1, u_2)$ belong to \mathbb{R}^2 and z_1, z_2 belong to S^1 , we have

$$\alpha_{(s,u)} f(z_1, z_2, \gamma_1, \gamma_2, \gamma_3) = f((e(-s_1), e(-s_2), 0, 0, 0)(z_1, z_2, \gamma_1, \gamma_2, \gamma_3)(e(u_1), e(u_2), 0, 0, 0)).$$

$$\text{Moreover, } \tilde{J} = -J \oplus J = \begin{pmatrix} 0 & \frac{\theta}{2} & 0 & 0 \\ -\frac{\theta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\theta}{2} \\ 0 & 0 & \frac{\theta}{2} & 0 \end{pmatrix}.$$

Hence, $\tilde{J}(s_1, s_2, t_1, t_2)^t = (\frac{\theta}{2}s_2, -\frac{\theta}{2}s_1, -\frac{\theta}{2}t_2, \frac{\theta}{2}t_1)^t$ where t stands for the transpose of a matrix.

Let $s = (s_1, s_2), t = (t_1, t_2), s' = (s'_1, s'_2), t' = (t'_1, t'_2)$ belong to \mathbb{R}^2 $u = (s, t), v = (s', t')$.

Lemma 4.4.10. $\{A_{(\gamma_1, \gamma_2, \gamma_3)}, B_{(\gamma_1, \gamma_2, \gamma_3)}\}$ are unitaries for the multiplication $\times_{\tilde{J}}$ and for $(\gamma_1, \gamma_2, \gamma_3)$ belonging to \mathbf{Z}_2^3 .

Proof : We give the proof for $A_{(1,0,1)}$, the proof for the rest being exactly similar.

$$\begin{aligned} & A_{(1,0,1)}^* \times_{\tilde{J}} A_{1,0,1}(u_1, u_2, 1, 0, 1) \\ &= \int \int \chi_{\tilde{J}(u)}(A_{(1,0,1)}^*)(u_1, u_2, 1, 0, 1) \chi_v(A_{1,0,1})(u_1, u_2, 1, 0, 1) \\ & \quad e(u.v) du dv \end{aligned}$$

$$\begin{aligned}
&= \int \int \chi_{(\frac{\theta}{2}s_2, -\frac{\theta}{2}s_1, -\frac{\theta}{2}t_2, \frac{\theta}{2}t_1)} A_{(1,0,1)}^*(u_1, u_2, 1, 0, 1) \chi_{(s'_1, s'_2, t'_1, t'_2)} A_{(1,0,1)}(u_1, \\
&\quad u_2, 1, 0, 1) e(u.v) dudv \\
&= \int \int A_{(1,0,1)}^*((e(-\frac{\theta}{2}s_2), e(\frac{\theta}{2}s_1))(u_1, u_2, 1, 0, 1)(e(-\frac{\theta}{2}t_2), e(\frac{\theta}{2}t_1))) \\
&\quad A_{(1,0,1)}((e(-s'_1), e(-s'_2))(u_1, u_2, 1, 0, 1)(e(t'_1), e(t'_2)))e(u.v) dudv
\end{aligned}$$

which by (4.4.1) equals

$$\begin{aligned}
&\int \int A_{(1,0,1)}^*(e(-\frac{\theta}{2}s_2)u_1e(-\frac{\theta}{2}t_1), e(\frac{\theta}{2}s_1)u_2e(-\frac{\theta}{2}t_2)) A_{(1,0,1)}(e(-s'_1)u_1e(-t'_2), \\
&\quad e(-s'_2)u_2e(t'_1))e(u.v) dudv \\
&= \int \dots \int e(\frac{\theta}{2}s_2)e(\frac{\theta}{2}t_1)e(-s'_1)e(-t'_2)e(s_1.s'_1 + t_1.t'_1 + s_2.s'_2 + t_2.t'_2) ds_1 ds'_1 ds_2 ds'_2 dt_1 \\
&\quad dt'_1 dt_2 dt'_2 \\
&= \int \int e(-s'_1)e(s_1.s'_1) ds_1 ds'_1 \int \int e(\frac{\theta}{2}s_2)e(s_2.s'_2) ds_2 ds'_2 \\
&\quad \int \int e(\frac{\theta}{2}t_1)e(t_1.t'_1) dt_1 dt'_1 \int \int e(-t'_2)e(t_2.t'_2) dt_2 dt'_2
\end{aligned}$$

which by Proposition 1.3.1 equals

$$1.1.1.1 = 1$$

Similarly,

$$A_{(1,0,1)} \times_{\tilde{J}} A_{1,0,1}^*(u_1, u_2, 1, 0, 1) = 1$$

□

Remark 4.4.11. *It can be easily checked that*

$$A_{\gamma_1, \gamma_2, \gamma_3}^* \times_{\tilde{J}} A_{\gamma_1, \gamma_2, \gamma_3}(u_1, u_2, \gamma'_1, \gamma'_2, \gamma'_3) = 0$$

if $(\gamma_1, \gamma_2, \gamma_3) \neq (\gamma'_1, \gamma'_2, \gamma'_3)$.

Similar is the case with $B_{\gamma_1, \gamma_2, \gamma_3}$.

Lemma 4.4.12. $A_{000} \times_{\tilde{J}} B_{000} = B_{000} \times_{\tilde{J}} A_{000}$.

Proof :

$$\begin{aligned}
& (A_{000} \times_{\tilde{J}} B_{000})((u_1, u_2, 0, 0, 0)) \\
&= \int \int \chi_{\tilde{J}_u}(A_{000})((u_1, u_2, 0, 0, 0)) \chi_v(B_{000})((u_1, u_2, 0, 0, 0)) e(u.v) du dv \\
&= \int \dots \int \chi_{\tilde{J}(s_1, s_2, t_1, t_2)} A_{000}((u_1, u_2, 0, 0, 0)) \chi_{(s'_1, s'_2, t'_1, t'_2)}(B_{000})((u_1, u_2, 0, 0, 0)) \\
&\quad e(u.v) ds dt ds' dt' \\
&= \int \dots \int \chi_{(\frac{\theta}{2}s_2, -\frac{\theta}{2}s_1, -\frac{\theta}{2}t_2, \frac{\theta}{2}t_1)} A_{000}((u_1, u_2, 0, 0, 0)) \chi_{(s'_1, s'_2, t'_1, t'_2)}(B_{000})((u_1, u_2, 0, 0, 0)) \\
&\quad e(u.v) ds dt ds' dt' \\
&= \int \dots \int A_{000}[(e(-\frac{\theta}{2}s_2), e(\frac{\theta}{2}s_1))(u_1, u_2, 0, 0, 0)(e(-\frac{\theta}{2}t_2), e(\frac{\theta}{2}t_1))] \\
&\quad B_{000}[(e(-s'_1), e(-s'_2))(u_1, u_2, 0, 0, 0)(e(t'_1), e(t'_2))] e(u.v) ds dt ds' dt'
\end{aligned}$$

which by (4.4.3) equals

$$\begin{aligned}
& \int \dots \int A_{000}[e(-\frac{\theta}{2}s_2)u_1e(-\frac{\theta}{2}t_2), e(\frac{\theta}{2}s_1)u_2e(\frac{\theta}{2}t_1), 0, 0, 0] B_{000}[(e(-s'_1)u_1e(t'_1)), (e(-s'_2) \\
&\quad u_2e(t'_2)), 0, 0, 0] e(u.v) ds dt ds' dt' \\
&= \int \dots \int e(-\frac{\theta}{2}s_2)u_1e(-\frac{\theta}{2}t_2)e(-s'_2)u_2e(t'_2)e(s_1s'_1)e(s_2s'_2)e(t_1t'_1)e(t_2t'_2) ds dt ds' dt' \\
&= u_1u_2 \int \int e(-\frac{\theta}{2}s_2)e(-s'_2)e(s_2s'_2) ds_2 ds'_2 \int \int e(-\frac{\theta}{2}t_2)e(t'_2)e(t_2t'_2) dt_2 dt'_2 \\
&\quad \int \int e(s_1s'_1) ds_1 ds'_1 \int \int e(t_1t'_1) dt_1 dt'_1
\end{aligned}$$

which by Corollary 1.3.4 and Proposition 1.3.1

$$\begin{aligned}
&= u_1u_2 e(-\frac{\theta}{2})e(\frac{\theta}{2}).1.1 \\
&= u_1u_2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& (B_{000} \times_{\tilde{J}} A_{000})((u_1, u_2, 0, 0, 0)) \\
&= \int \dots \int B_{000}[(e(-\frac{\theta}{2}s_2)u_1e(-\frac{\theta}{2}t_2)), (e(\frac{\theta}{2}s_1)u_2e(\frac{\theta}{2}t_1), 0, 0, 0)] \\
&\quad A_{000}[(e(-s'_1)u_1e(t'_1)), (e(-s'_2)u_2e(t'_2)), 0, 0, 0)] e(u.v) ds dt ds' dt' \\
&= \int \dots \int e(\frac{\theta}{2}s_1)u_2e(\frac{\theta}{2}t_1)e(-s'_1)u_1e(t'_1)e(u.v) ds dt ds' dt' \\
&= u_1u_2 \int \int e(\frac{\theta}{2}s_1)e(-s'_1)e(s_1s'_1) ds_1 ds'_1 \int \int e(\frac{\theta}{2}t_1)e(t'_1)e(t_1t'_1) dt_1 dt'_1 \\
&\quad \int \int e(s_2s'_2) ds_2 ds'_2 \int \int e(t_2t'_2) dt_2 dt'_2,
\end{aligned}$$

which by Corollary 1.3.4 and Proposition 1.3.1

$$= u_1 u_2.$$

Therefore, $A_{000} \times_{\tilde{J}} B_{000} = B_{000} \times_{\tilde{J}} A_{000}$. □

Lemma 4.4.13. $A_{001} \times_{\tilde{J}} B_{001} = e(-2\theta)B_{001} \times_{\tilde{J}} A_{001}$

Proof : Proceeding exactly as in the previous Lemma, we obtain

$$\begin{aligned} & (A_{001} \times_{\tilde{J}} B_{001})(u_1, u_2, 0, 0, 1) \\ &= \int \dots \int A_{001}[(e(-\frac{\theta}{2}s_2), e(\frac{\theta}{2}s_1))(u_1, u_2, 0, 0, 1)(e(-\frac{\theta}{2}t_2), e(\frac{\theta}{2}t_1))] \\ & \quad B_{001}[(e(-s'_1), e(-s'_2))(u_1, u_2, 0, 0, 1)(e(t'_1), e(t'_2))]e(u.v)dsdt ds' dt', \end{aligned}$$

which by (4.4.2) equals

$$\begin{aligned} & \int \dots \int A_{001}[e(-\frac{\theta}{2}s_2)u_1e(\frac{\theta}{2}t_1), e(\frac{\theta}{2}s_1)u_2e(-\frac{\theta}{2}t_2), 0, 0, 1] \\ & \quad B_{001} [e(-s'_1)u_1e(t'_2), e(-s'_2)u_2e(t'_1), 0, 0, 1]e(u.v)dsdt ds' dt' \\ &= u_1 u_2 \int \dots \int e(-\frac{\theta}{2}s_2)e(\frac{\theta}{2}t_1)e(-s'_2)e(t'_1)e(u.v)dsdt ds' dt' \\ &= u_1 u_2 \int \int e(-\frac{\theta}{2}s_2)e(-s'_2)e(s_2.s'_2)ds_2 ds'_2 \int \int e(\frac{\theta}{2}t_1)e(t'_1)e(t_1.t'_1)dt_1 dt'_1 \\ & \quad \int \int e(s_1.s'_1)ds_1 ds'_1 \int \int e(t_2.t'_2)dt_2 dt'_2 \end{aligned}$$

which by Corollary 1.3.4 and Proposition 1.3.1 equals

$$\begin{aligned} & u_1 u_2 e(-\frac{\theta}{2})e(-\frac{\theta}{2}) \\ &= u_1 u_2 e(-\theta). \end{aligned}$$

Similarly, using Corollary 1.3.4 and Proposition 1.3.1

$$\begin{aligned} & (B_{001} \times_{\tilde{J}} A_{001})(u_1, u_2, 0, 0, 1) \\ &= \int \dots \int B_{001}[e(-\frac{\theta}{2}s_2)u_1e(\frac{\theta}{2}t_1), e(\frac{\theta}{2}s_1)u_2e(-\frac{\theta}{2}t_2), 0, 0, 1] \\ & \quad A_{001}[e(-s'_1)u_1e(t'_2), e(-s'_2)u_2e(t'_1), 0, 0, 1]e(u.v)dsdt ds' dt' \\ &= u_1 u_2 \int \dots \int e(\frac{\theta}{2}s_1)e(-\frac{\theta}{2}t_2)e(-s'_1)e(t'_2)e(s_1.s'_1 + s_2.s'_2 + t_1.t'_1 + t_2.t'_2)dsdt ds' dt' \\ &= u_1 u_2 e(\frac{\theta}{2})e(\frac{\theta}{2}) \\ &= u_1 u_2 e(\theta). \end{aligned}$$

This proves the Lemma. □

Remark 4.4.14. *Proceeding similarly, one can prove $A_{0,1,1} \times_{\tilde{J}} B_{0,1,1} = B_{0,1,1} \times_{\tilde{J}} A_{0,1,1}$, $A_{1,1,0} \times_{\tilde{J}} B_{1,1,0} = B_{1,1,0} \times_{\tilde{J}} A_{1,1,0}$, $A_{1,0,1} \times_{\tilde{J}} B_{1,0,1} = B_{1,0,1} \times_{\tilde{J}} A_{1,0,1}$ and $A_{0,0,1} \times_{\tilde{J}} B_{0,0,1} = e(-2\theta)B_{0,0,1} \times_{\tilde{J}} A_{0,0,1}$, $A_{1,0,0} \times_{\tilde{J}} B_{1,0,0} = e(-2\theta)B_{1,0,0} \times_{\tilde{J}} A_{1,0,0}$, $A_{1,1,1} \times_{\tilde{J}} B_{1,1,1} = e(-2\theta)B_{1,1,1} \times_{\tilde{J}} A_{1,1,1}$.*

Let us now consider a C^* algebra \mathcal{B} , which has eight direct summands, four of which are isomorphic with the commutative algebra $C(\mathbb{T}^2)$, and the other four are irrational rotation algebras. More precisely, we take

$$\mathcal{B} = \oplus_{k=1}^8 C^*(U_{k1}, U_{k2}),$$

where for odd k , U_{k1}, U_{k2} are the two commuting unitary generators of $C(\mathbb{T}^2)$, and for even k , $U_{k1}U_{k2} = e(2\pi i\theta)U_{k2}U_{k1}$, that is they generate $\mathcal{A}_{2\theta}$.

Now we are in a position to describe $QISO^{\mathcal{L}}(\mathcal{A}_\theta)$ explicitly.

Theorem 4.4.15. *$QISO^{\mathcal{L}}(\mathcal{A}_\theta)$ is isomorphic with $\mathcal{B} = C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta} \oplus C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta} \oplus C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta} \oplus C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta}$.*

Proof : Define $\phi : \mathcal{B} \rightarrow QISO^{\mathcal{L}}(\mathcal{A}_\theta)$ by

$$\begin{aligned} \phi(U_{11}) &= A_{000}, \phi(U_{12}) = B_{000}, \phi(U_{21}) = A_{010}, \phi(U_{22}) = A_{010}, \phi(U_{31}) = A_{011}, \\ \phi(U_{32}) &= B_{011}, \phi(U_{41}) = A_{001}, \phi(U_{42}) = B_{001}, \phi(U_{51}) = A_{110}, \phi(U_{52}) = B_{110}, \\ \phi(U_{61}) &= A_{111}, \phi(U_{62}) = B_{111}, \phi(U_{71}) = A_{101}, \phi(U_{72}) = B_{101}, \phi(U_{81}) = A_{100}, \\ \phi(U_{82}) &= B_{100}. \end{aligned}$$

From Lemma 4.4.10 - Lemma 4.4.13 and Remark 4.4.14, it is clear that ϕ is indeed a $*$ isomorphism. □

Remark 4.4.16. *In particular, we note that if θ is taken to be $1/2$, then we have a commutative compact quantum group as the quantum isometry group of a noncommutative C^* algebra.*

The case of θ deformed spheres

We can apply Theorem 4.4.7 on S_θ^n which are obtained by Rieffel deformation of $C(S^n)$ as described in section 1.3. We will consider the isospectral deformation of the classical spectral triple on $C(S^n)$. Since we have proved in Theorem 2.2.2 that $QISO^{\mathcal{L}}(S^n) \cong C(O(n))$, it will follow that $QISO^{\mathcal{L}}(S_\theta^n) \cong O_\theta(n)$, where $O_\theta(n)$ is the compact quantum group discussed in subsection 1.3.1 as the θ -deformation of $C(O(n))$.

Chapter 5

Quantum isometry groups of the Podles spheres

In this chapter, we compute quantum group of orientation preserving isometries for spectral triples on the Podles spheres $S_{\mu,c}^2$. We do it for two different families of spectral triples, one constructed by Dabrowski et al in [24] and the other by Chakraborty and Pal in [14] for $c > 0$. We get completely different kinds of quantum group of orientation preserving isometries for the two families; for the former, it is $SO_\mu(3)$, whereas, for the latter it is $C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$ where \mathbf{Z}^∞ denotes countably infinite copies of the group of integers.

5.1 The Podles Spheres

The Podles spheres were discovered in [43]. We will also need the equivalent descriptions of the Podles spheres as given in [24], [37] and [50]. Hence, we give the descriptions one by one.

5.1.1 The original definition by Podles

Let c belongs to \mathbb{R} . The Podles' sphere $S_{\mu,c}^2$ is the universal C^* algebra generated by elements e_{-1}, e_0, e_1 such that :

$$\begin{aligned} e_i^* &= e_{-i}, \quad i = -1, 0, 1, \\ (1 + \mu^2)(e_{-1}e_1 + \mu^{-2}e_1e_{-1}) + e_0^2 &= ((1 + \mu^2)^2\mu^{-2}c + 1)I, \\ e_0e_{-1} - \mu^2e_{-1}e_0 &= (1 - \mu^2)e_{-1}, \\ (1 + \mu^2)(e_{-1}e_1 - e_1e_{-1}) + (1 - \mu^2)e_0^2 &= (1 - \mu^2)e_0, \end{aligned}$$

$$e_1 e_0 - \mu^2 e_0 e_1 = (1 - \mu^2) e_1.$$

Let

$$B = e_1, \quad A = (1 + \mu^2)^{-1} (1 - e_0). \quad (5.1.1)$$

Then we have an alternate description of the Podles spheres, that is, the universal C^* algebra generated by elements A and B satisfying the relations:

$$A^* = A, \quad AB = \mu^{-2} BA,$$

$$B^* B = A - A^2 + cI, \quad BB^* = \mu^2 A - \mu^4 A^2 + cI.$$

These are the relations which we are going to exploit for getting homomorphism conditions while computing the quantum group of orientation preserving isometries.

Notation : We will denote the co-ordinate $*$ algebra of $S_{\mu,c}^2$ by $\mathcal{O}(S_{\mu,c}^2)$.

5.1.2 The Podles spheres as in [24]

We take μ in $(0, 1)$ and t in $(0, 1]$. For n belonging to \mathbb{R} , let $[n]_\mu = \frac{\mu^n - \mu^{-n}}{\mu - \mu^{-1}}$.

Then $S_{\mu,c}^2$ be the universal C^* algebra generated by elements x_{-1}, x_0, x_1 satisfying the relations:

$$x_{-1}(x_0 - t) = \mu^2(x_0 - t)x_{-1}, \quad (5.1.2)$$

$$x_1(x_0 - t) = \mu^{-2}(x_0 - t)x_1, \quad (5.1.3)$$

$$- [2]x_{-1}x_1 + (\mu^2 x_0 + t)(x_0 - t) = [2]^2(1 - t), \quad (5.1.4)$$

$$- [2]x_1x_{-1} + (\mu^{-2} x_0 + t)(x_0 - t) = [2]^2(1 - t), \quad (5.1.5)$$

where $c = t^{-1} - t, t > 0$.

The involution on $S_{\mu,c}^2$ is given by

$$x_{-1}^* = -\mu^{-1}x_1, \quad x_0^* = x_0. \quad (5.1.6)$$

Setting

$$A = \frac{1 - t^{-1}x_0}{1 + \mu^2}, \quad B = \mu(1 + \mu^2)^{-\frac{1}{2}}t^{-1}x_{-1}, \quad (5.1.7)$$

one obtains ([24]) that $S_{\mu,c}^2$ is the same as the Podles' sphere as in [43].

5.1.3 The Podles spheres as in [37]

This description is taken from [37], page 124.

Let α', β be elements of \mathbb{C} such that $(\alpha', \beta) \neq (0, 0)$. We denote by $\chi_{q,\alpha',\beta}$ the algebra with three generators x_{-1}, x_0, x_1 and the following defining relations:

$$x_0^2 - qx_1x_{-1} - q^{-1}x_{-1}x_1 = \beta.1, \quad (5.1.8)$$

$$(1 - q^2)x_0^2 + qx_{-1}x_1 - qx_1x_{-1} = (1 - q^2)\alpha'x_0, \quad (5.1.9)$$

$$x_{-1}x_0 - q^2x_0x_{-1} = (1 - q^2)\alpha'x_{-1}, \quad (5.1.10)$$

$$x_0x_1 - q^2x_1x_0 = (1 - q^2)\alpha'x_1. \quad (5.1.11)$$

Let $\rho^2 = \alpha^2(\beta - \alpha^2)^{-1}$.

Then for q and ρ real, the involution is defined by $x_{-1}^* = -q^{-1}x_1$, $x_0^* = x_0$, $x_1^* = -qx_{-1}$.

Moreover, from page 125 of [37], it follows that for ρ belonging to \mathbb{C} , $\chi_{q,\alpha',\beta}$ can be realized as a $*$ subalgebra of $SU_\mu(2)$ via:

$$x_{-1} = (1 + q^2)^{-\frac{1}{2}}a^2 + \rho(1 + q^{-2})^{\frac{1}{2}}ac - q(1 + q^2)^{-\frac{1}{2}}c^2, \quad (5.1.12)$$

$$x_0 = ba + \rho(1 + (q + q^{-1})bc) - cd, \quad (5.1.13)$$

$$x_1 = (1 + q^2)^{-\frac{1}{2}}b^2 + \rho(1 + q^2)^{\frac{1}{2}}db - q(1 + q^2)^{-\frac{1}{2}}d^2. \quad (5.1.14)$$

where a, b, c, d are as in subsection 1.2.5.

Proposition 5.1.1. $S_{\mu,c}^2$ as defined above is the same as $\chi_{q,\alpha',\beta}$ with $q = \mu$, $\alpha' = t$, $\beta = t^2 + \mu^{-2}(\mu^2 + 1)^2(1 - t)$.

Proof : We note that $[2]_\mu = \frac{\mu^2 - \mu^{-2}}{\mu - \mu^{-1}} = \frac{\mu^2 + 1}{\mu}$.

From (5.1.5), we have

$$- [2]\mu^2x_1x_{-1} + x_0^2 + (t\mu^2 - t)x_0 - \mu^2t^2 = [2]^2\mu^2(1 - t). \quad (5.1.15)$$

Adding (5.1.4) with (5.1.15), we obtain

$$- [2]x_{-1}x_1 - [2]\mu^2x_1x_{-1} + (\mu^2 + 1)x_0^2 - (1 + \mu^2)t^2 = [2]^2(1 - t)(1 + \mu^2).$$

Using $[2]_\mu = \frac{1 + \mu^2}{\mu}$, we get this to be

$$- \frac{1 + \mu^2}{\mu}x_{-1}x_1 - \frac{1 + \mu^2}{\mu}\mu^2x_1x_{-1} + (1 + \mu^2)x_0^2 - (1 + \mu^2)t^2 = \frac{(1 + \mu^2)^2}{\mu^2}(1 - t)(1 + \mu^2).$$

Hence,

$$x_0^2 - \mu x_1x_{-1} - \mu^{-1}x_{-1}x_1 = \mu^{-2}(1 + \mu^2)^2(1 - t) + t^2 = \beta.1.$$

Thus, we have derived (5.1.8).

Multiplying (5.1.4) by μ^2 , we have

$$- [2]\mu^2 x_{-1}x_1 + \mu^4 x_0^2 + t(1 - \mu^2)\mu^2 x_0 - t^2\mu^2 = [2]^2\mu^2(1 - t). \quad (5.1.16)$$

Subtracting (5.1.15) from (5.1.16) gives

$$-\frac{\mu^2 + 1}{\mu}\mu^2 x_{-1}x_1 + (\mu^4 - 1)x_0^2 + t(1 - \mu^2)\mu^2 x_0 + \frac{\mu^2 + 1}{\mu}\mu^2 x_1x_{-1} - t(\mu^2 - 1)x_0 = 0,$$

and hence,

$$(1 - \mu^2)x_0^2 + \mu x_{-1}x_1 - \mu x_1x_{-1} = t(1 - \mu^2)x_0.$$

which proves (5.1.9).

Next, (5.1.2) gives

$$x_{-1}x_0 - \mu^2 x_0x_{-1} = (1 - \mu^2)tx_{-1}$$

which is (5.1.10).

Finally, (5.1.3) gives $\mu^2 x_1x_0 - \mu^2 tx_1 = x_0x_1 - tx_1$, that is, (5.1.11) is obtained.

Thus, we obtain the relations of $\chi_{q,\alpha',\beta}$ from the relations of $S_{\mu,c}^2$ for $q = \mu$, $\alpha' = t$, $\beta = t^2 + \mu^{-2}(\mu^2 + 1)^2(1 - t)$.

Similarly, (5.1.10) is same as (5.1.2), (5.1.11) is same as (5.1.3). Subtracting (5.1.9) from (5.1.8) gives (5.1.4) and adding (5.1.8) with μ^{-2} times (5.1.9) gives (5.1.5). Thus, we get back the relations of $S_{\mu,c}^2$ from the relations of $\chi_{q,\alpha',\beta}$. \square

\square

Thus, combining (5.1.12) - (5.1.14) with the correspondence (1.2.26) and using Proposition 5.1.1, we have expressions of x_{-1} , x_0 , x_1 in terms of $SU_\mu(2)$ elements:

$$x_{-1} = \frac{\mu\alpha^2 + \rho(1 + \mu^2)\alpha\gamma - \mu^2\gamma^2}{\mu(1 + \mu^2)^{\frac{1}{2}}}, \quad (5.1.17)$$

$$x_0 = -\mu\gamma^*\alpha + \rho(1 - (1 + \mu^2)\gamma^*\gamma) - \gamma\alpha^*, \quad (5.1.18)$$

$$x_1 = \frac{\mu^2\gamma^{*2} - \rho\mu(1 + \mu^2)\alpha^*\gamma^* - \mu\alpha^{*2}}{(1 + \mu^2)^{\frac{1}{2}}}, \quad (5.1.19)$$

where $\rho^2 = \frac{\mu^2 t^2}{(\mu^2 + 1)^2(1 - t)}$.

5.1.4 The description as in [50]

We need the quantum group $\mathcal{U}_\mu(su(2))$ for this description.

Let

$$X_c = \mu^{\frac{1}{2}}(\mu^{-1} - \mu)^{-1}c^{-\frac{1}{2}}(1 - K^2) + EK + \mu FK \quad \text{for all } c \in (0, \infty),$$

$$X_0 = 1 - K^2.$$

One has $\Delta(X_c) = 1 \otimes X_c + X_c \otimes K^2$.

Then (as in Page 9, [50]) we have the following :

$$\mathcal{O}(S_{\mu,c}^2) = \{x \in \mathcal{O}(SU_\mu(2)) : x \triangleleft X_c = 0\}$$

where \triangleleft is as in subsection 1.2.5. The following is a basis of the vector space $\mathcal{O}(S_{\mu,c}^2)$:

$$\{A^k, A^k B^l, A^k B^{*m}, k \geq 0, l, m > 0\}.$$

So, any element of $\mathcal{O}(S_{\mu,c}^2)$ can be written as a *finite* linear combination of elements of the form $A^k, A^k B^l, A^k B^{*l}$.

Let ψ be the densely defined linear map on $L^2(SU_\mu(2))$ defined by $\psi(x) = x \triangleleft X_c$.

From [51], (Page 5), we recall that $v_{j,k}^l \triangleleft E = \mu \alpha_{k-1}^l v_{j,k-1}^l$, $v_{j,k}^l \triangleleft F = \frac{\alpha_k^l}{\mu} v_{j,k+1}^l$, $v_{j,k}^l \triangleleft K = \mu^k v_{j,k}^l$ for some constants α_j^l .

Lemma 5.1.2. *The map ψ is closable and we have $\overline{S_{\mu,c}^2} \subseteq \text{Ker}(\overline{\psi})$ where $\overline{\psi}$ is the closed extension of ψ and $\overline{S_{\mu,c}^2}$ denotes the Hilbert subspace generated by $S_{\mu,c}^2$ in $L^2(SU_\mu(2))$ (the G.N.S space corresponding to the Haar state on $SU_\mu(2)$). Moreover, $\mathcal{O}(S_{\mu,c}^2) = \mathcal{O}(SU_\mu(2)) \cap \text{Ker}(\overline{\psi}) = \mathcal{O}(SU_\mu(2)) \cap \text{Ker}(\psi)$.*

Proof : We note that $v_{j,k}^l \triangleleft (1 - K^2) = (1 - \mu^{2k})v_{j,k}^l$, $v_{j,k}^l \triangleleft EK = \mu^k \alpha_{k-1}^l v_{j,k-1}^l$, $v_{j,k}^l \triangleleft FK = \mu^k \alpha_k^l v_{j,k+1}^l$. Hence, $v_{j,k}^l$ belongs to $\text{Dom}(\psi)$ with $(\psi^*)(v_{j,k}^l) = \mu^{\frac{1}{2}}(\mu^{-1} - \mu)^{-1}c^{-\frac{1}{2}}$.

$(1 - \mu^{2k})v_{j,k}^l + \mu^k \alpha_{k-1}^l v_{j,k+1}^l + \mu^{k+1} \alpha_k^l v_{j,k-1}^l$. Hence, $\mathcal{O}(SU_\mu(2)) \subseteq \text{Dom}(\psi^*)$ implying that ψ is closable, hence $\text{Ker}(\overline{\psi})$ is closed. The lemma now follows from the observation that $\mathcal{O}(S_{\mu,c}^2) = \text{Ker}(\psi) \subseteq \text{Ker}(\overline{\psi})$. \square

5.1.5 Haar functional on the Podles spheres

We recall from [50], page- 33 that for all bounded complex Borel function f on $\sigma(A)$,

$$h(f(A)) = \gamma_+ \sum_{n=0}^{\infty} f(\lambda_+ \mu^{2n}) \mu^{2n} + \gamma_- \sum_{n=0}^{\infty} f(\lambda_- \mu^{2n}) \mu^{2n}. \quad (5.1.20)$$

where $\lambda_+ = \frac{1}{2} + (c + \frac{1}{4})^{\frac{1}{2}}$, $\lambda_- = \frac{1}{2} - (c + \frac{1}{4})^{\frac{1}{2}}$, $\gamma_+ = (1 - \mu^2)\lambda_+(\lambda_+ - \lambda_-)^{-1}$, $\gamma_- = (1 - \mu^2)\lambda_-(\lambda_- - \lambda_+)^{-1}$.

Lemma 5.1.3. $\{x_{-1}, x_0, x_1\}$ is a set of orthogonal vectors.

Proof : From (5.1.17) and (5.1.18), we note that $x_{-1}^*x_0$ belongs to $\text{span}\{\alpha^{*2}\gamma^*\alpha, \alpha^{*2}, \alpha^{*2}\gamma^*\gamma, \alpha^{*2}\gamma\alpha^*, \gamma^*\alpha^*\gamma^*\alpha, \gamma^*\alpha^*, \gamma^*\alpha^*\gamma^*\gamma, \gamma^*\alpha^*\gamma\alpha^*, \gamma^{*3}\alpha, \gamma^{*2}, \gamma^{*3}\gamma, \gamma^{*2}\gamma\alpha^*\}$.

Further, using (1.2.10) - (1.2.14), we note that this span equals $\text{span}\{\alpha^*\gamma^*, \alpha^*\gamma^{*2}\gamma, \alpha^{*2}, \alpha^{*2}\gamma^*\gamma, \alpha^{*3}\gamma, \gamma^{*2}, \gamma^{*3}\gamma, \alpha^*\gamma^*, \alpha\gamma^{*3}, \gamma^{*3}\}$.

Then $h(x_{-1}^*x_0) = 0$ follows by using (1.2.15). Similarly, one can prove $h(x_0^*x_1) = h(x_1^*x_{-1}) = 0$. \square

Lemma 5.1.4.

$$h(A) = \frac{1}{1 + \mu^2}, \quad h(A^2) = \frac{(1 - \mu^2)(\lambda_+^3 - \lambda_-^3)}{(\lambda_+ - \lambda_-)(1 - \mu^6)}.$$

Proof : Recalling (5.1.20), we have

$$\begin{aligned} h(A) &= \gamma_+ \sum_{n=0}^{\infty} \lambda_+ \mu^{4n} + \gamma_- \sum_{n=0}^{\infty} \lambda_- \mu^{4n} \\ &= \frac{(1 - \mu^2)(\lambda_+^2 - \lambda_-^2)}{(\lambda_+ - \lambda_-)(1 - \mu^4)} \\ &= \frac{\lambda_+ + \lambda_-}{1 + \mu^2} \\ &= \frac{1}{1 + \mu^2}. \end{aligned}$$

Similarly, putting $f(x) = x^2$ in (5.1.20), we have

$$\begin{aligned} h(A^2) &= \gamma_+ \sum_{n=0}^{\infty} \lambda_+^2 \mu^{6n} + \gamma_- \sum_{n=0}^{\infty} \lambda_-^2 \mu^{6n} \\ &= \frac{(1 - \mu^2)(\lambda_+^3 - \lambda_-^3)}{(\lambda_+ - \lambda_-)(1 - \mu^6)}. \end{aligned}$$

\square

Proposition 5.1.5. $h(x_{-1}^*x_{-1}) = h(x_0^*x_0) = h(x_1^*x_1) = t^2(1 - \mu^2)(1 - \mu^6)^{-1}[\mu^2 + t^{-1}(\mu^4 + 2\mu^2 + 1) + t(-\mu^4 - 2\mu^2 - 1)]$.

Proof :

From (5.1.7) we have $x_{-1}^*x_{-1} = \frac{t^2(1+\mu^2)}{\mu^2}B^*B$ and hence, using Lemma 5.1.4, we

obtain

$$\begin{aligned}
& h(x_{-1}^* x_{-1}) \\
&= \frac{t^2(1+\mu^2)}{\mu^2} [h(A) - h(A^2) + (t^{-1} - t).1] \\
&= \frac{1 - \mu^6 - (1 - \mu^4)(t^{-1} - t + 1) + (t^{-1} - t)(1 + \mu^2)(1 - \mu^6)}{(1 + \mu^2)(1 - \mu^6)},
\end{aligned}$$

from which we get the desired result.

Similarly, the second equality is derived from $h(x_0^* x_0) = t^2 - 2t^2(1 + \mu^2)h(A) + (1 + \mu^2)^2 t^2 h(A^2)$.

From (5.1.6), $x_1 = -\mu x_{-1}^*$ and hence, $x_1^* x_1 = t^2(1 + \mu^2)(\mu^2 A - \mu^4 A^2 + c.I)$, from which $h(x_1^* x_1)$ is obtained and can be shown to be equal to the same value as $h(x_{-1}^* x_{-1}) = h(x_0^* x_0)$.

□

5.2 Spectral triples on the Podles spheres

5.2.1 The spectral triple as in [24]

We now recall the spectral triples on $S_{\mu c}^2$ discussed in [24].

Let $s = -c^{-\frac{1}{2}}\lambda_-$, $\lambda_{\pm} = \frac{1}{2} \pm (c + \frac{1}{4})^{\frac{1}{2}}$.

For all j belonging to $\frac{1}{2}\mathbb{N}$,

$$u_j = (\alpha^* - s\gamma^*)(\alpha^* - \mu^{-1}s\gamma^*)\dots(\alpha^* - \mu^{-2j+1}s\gamma^*),$$

$$w_j = (\alpha - \mu s\gamma)(\alpha - \mu^2 s\gamma)\dots(\alpha - \mu^{2j} s\gamma),$$

$$u_{-j} = E^{2j} \triangleright w_j,$$

$$u_0 = w_0 = 1,$$

$$y_1 = (1 + \mu^{-2})^{\frac{1}{2}}(c^{\frac{1}{2}}\mu^2\gamma^{*2} - \mu\gamma^*\alpha^* - \mu c^{\frac{1}{2}}\alpha^{*2}),$$

$$N_{kj}^l = \|F^{l-k} \triangleright (y_1^{l-|j|} u_j)\|^{-1}.$$

Define

$$v_{k,j}^l = N_{k,j}^l F^{l-k} \triangleright (y_1^{l-|j|} u_j), \quad l \in \frac{1}{2}\mathbb{N}_0, \quad j, k = -l, -l+1, \dots, l. \quad (5.2.1)$$

Let \mathcal{M}_N be the Hilbert subspace of $L^2(SU_{\mu}(2))$ with the orthonormal basis $\{v_{m,N}^l : l = |N|, |N|+1, \dots, m = -l, \dots, l\}$.

Set

$$\mathcal{H} = \mathcal{M}_{-\frac{1}{2}} \oplus \mathcal{M}_{\frac{1}{2}},$$

and define a representation π of $S_{\mu,c}^2$ on \mathcal{H} by

$$\pi(x_i)v_{m,N}^l = \alpha_i^-(l, m; N)v_{m+i,N}^{l-1} + \alpha_i^0(l, m; N)v_{m+i,N}^l + \alpha_i^+(l, m; N)v_{m+i,N}^{l+1}, \quad (5.2.2)$$

where α_i^- , α_i^0 , α_i^+ are some constants.

We will often identify $\pi(S_{\mu,c}^2)$ with $S_{\mu,c}^2$.

Finally by Proposition 7.2 of [24], the following Dirac operator D gives a spectral triple $(\mathcal{O}(S_{\mu,c}^2), \mathcal{H}, D)$ which we are going to work with :

$$D(v_{m,\pm\frac{1}{2}}^l) = (c_1 l + c_2)v_{m,\mp\frac{1}{2}}^l, \quad (5.2.3)$$

where c_1, c_2 are elements of \mathbb{R} , $c_1 \neq 0$.

5.2.2 $SU_\mu(2)$ equivariance of the spectral triple

From [24], we see that the vector spaces $\nu_{\pm\frac{1}{2}}^l = \text{span}\{v_{m,\pm\frac{1}{2}}^l : m = -l, \dots, l\}$ are $(2l+1)$ dimensional Hilbert spaces on which the $SU_\mu(2)$ representation is unitarily equivalent to the standard l th unitary irreducible representation of $SU_\mu(2)$, that is, if the representation is denoted by U_0 , then $U_0(v_{i,\pm\frac{1}{2}}^l) = \sum v_{j,\pm\frac{1}{2}}^l \otimes t_{i,j}^l$ where $t_{i,j}^l$ denotes the matrix elements in the l th unitary irreducible representation of $SU_\mu(2)$.

We now recall Theorem 3.5 of [32].

Proposition 5.2.1. *Let R_0 be an operator on \mathcal{H} defined by $R_0(v_{i,\pm\frac{1}{2}}^n) = \mu^{-2i\mp 1}v_{i,\pm\frac{1}{2}}^n$. Then $\text{Tr}(R_0 e^{-tD^2}) < \infty$ (for all $t > 0$) and one has*

$$(\tau_{R_0} \otimes \text{id})(\widetilde{U_0}(x \otimes 1)\widetilde{U_0}^*) = \tau_{R_0}(x).1,$$

for all x in $\mathcal{B}(\mathcal{H})$, where $\tau_{R_0}(x) = \text{Tr}(xR_0 e^{-tD^2})$.

We define a positive, unbounded operator R on \mathcal{H} by $R(v_{i,\pm\frac{1}{2}}^n) = \mu^{-2i}v_{i,\pm\frac{1}{2}}^n$.

Proposition 5.2.2. *α_{U_0} preserves the R -twisted volume. In particular, for x in $\pi(S_{\mu,c}^2)$ and $t > 0$, we have $h(x) = \frac{\tau_R(x)}{\tau_R(1)}$, where $\tau_R(x) := \text{Tr}(xR e^{-tD^2})$, and h denotes the restriction of the Haar state of $SU_\mu(2)$ to the subalgebra $S_{\mu,c}^2$, which is the unique $SU_\mu(2)$ -invariant state on $S_{\mu,c}^2$.*

Proof : It is enough to prove that τ_R is α_{U_0} -invariant. Let us denote by $P_{\frac{1}{2}}, P_{-\frac{1}{2}}$ the projections onto the closed subspaces generated by $\{v_{i,\frac{1}{2}}^l\}$ and $\{v_{i,-\frac{1}{2}}^l\}$ respectively. Moreover, let τ_\pm be the functionals defined by $\tau_\pm(x) = \text{Tr}(xR_0 P_{\pm\frac{1}{2}} e^{-tD^2})$. Now observing that R_0 , e^{-tD^2} and U_0 commute with $P_{\pm\frac{1}{2}}$ and using Proposition 5.2.1, we have, for

x belonging to $\mathcal{B}(\mathcal{H})$,

$$\begin{aligned}
& (\tau_{\pm} \otimes \text{id})(\alpha_{U_0}(x)) \\
&= (\text{Tr} \otimes \text{id})(\widetilde{U}_0(x \otimes 1)\widetilde{U}_0^*(R_0 P_{\pm\frac{1}{2}} e^{-tD^2} \otimes \text{id})) \\
&= (\text{Tr} \otimes \text{id})(\widetilde{U}_0(x P_{\pm\frac{1}{2}} \otimes 1)\widetilde{U}_0^*(R_0 e^{-tD^2} \otimes \text{id})) \\
&= (\tau_{R_0} \otimes \text{id})\alpha_{U_0}(x P_{\pm\frac{1}{2}}) \\
&= \tau_{R_0}(x P_{\pm\frac{1}{2}}) \\
&= \tau_{\pm}(x).1,
\end{aligned}$$

that is τ_{\pm} are α_{U_0} -invariant.

Thus, $x \mapsto \text{Tr}(x R_0 P_{\pm\frac{1}{2}} e^{-tD^2})$ is invariant under α_{U_0} . Moreover, since we have $R P_{\pm\frac{1}{2}} = \mu^{\pm} R_0 P_{\pm\frac{1}{2}}$, the functional τ_R coincides with $\mu^{-1}\tau_+ + \mu\tau_-$, hence is α_{U_0} -invariant. \square

Theorem 5.2.3. $(SU_{\mu}(2), U_0)$ is an object in $\mathbf{Q}'_R(D)$.

Proof :

$$\begin{aligned}
(D \otimes \text{id})U_0(v_{i,\pm\frac{1}{2}}^l) &= (D \otimes \text{id}) \sum v_{j,\pm\frac{1}{2}}^l \otimes t_{i,j}^l \\
&= (c_1 l + c_2) \sum v_{j,\mp\frac{1}{2}}^l \otimes t_{i,j}^l \\
&= (c_1 l + c_2) U_0(v_{i,\mp\frac{1}{2}}^l) \\
&= U_0 D(v_{i,\pm\frac{1}{2}}^l).
\end{aligned}$$

Thus, the above spectral triple is equivariant w.r.t. the representation U_0 and it preserves τ_R by Proposition 5.2.2, which completes the proof. \square

5.2.3 The CQG $SO_{\mu}(3)$ and its action on the Podles sphere

Here we recall the CQG $SO_{\mu}(3)$ as described in [44].

$SO_{\mu}(3)$ is the universal unital C^* algebra generated by elements M, N, G, C, L satisfying :

$$\left\{ \begin{array}{l} L^*L = (I - N)(I - \mu^{-2}N), LL^* = (I - \mu^2N)(I - \mu^4N), G^*G = GG^* = N^2, \\ M^*M = N - N^2, MM^* = \mu^2N - \mu^4N^2, C^*C = N - N^2, \\ CC^* = \mu^2N - \mu^4N^2, LN = \mu^4NL, GN = NG, \\ MN = \mu^2NM, CN = \mu^2NC, LG = \mu^4GL, \\ LM = \mu^2ML, MG = \mu^2GM, CM = MC, \\ LG^* = \mu^4G^*L, M^2 = \mu^{-1}LG, M^*L = \mu^{-1}(I - N)C, N^* = N. \end{array} \right. \quad (5.2.4)$$

This CQG can be identified with a Woronowicz subalgebra of $SU_\mu(2)$ by taking:

$$N = \gamma^* \gamma, \quad M = \alpha \gamma, \quad C = \alpha \gamma^*, \quad G = \gamma^2, \quad L = \alpha^2,$$

where α, γ are as in subsection 1.2.4.

The canonical action of $SU_\mu(2)$ on $S_{\mu,c}^2$, that is the action obtained by restricting the coproduct of $SU_\mu(2)$ to the subalgebra $S_{\mu,c}^2$, is actually a faithful action of $SO_\mu(3)$.

With respect to the ordered basis $\{x_{-1}, x_0, x_1\}$, this action on the subspace generated by them is given by the following $SO_\mu(3)$ -valued 3×3 -matrix ([37]):

$$\begin{pmatrix} a^2 & \frac{(1+q^2)^{\frac{1}{2}}}{q} ab & b^2 \\ \frac{(1+q^2)^{\frac{1}{2}}}{q} ac & I + (q + q^{-1})bc & \frac{(1+q^2)^{\frac{1}{2}}}{q} bd \\ c^2 & \frac{(1+q^2)^{\frac{1}{2}}}{q} cd & d^2 \end{pmatrix}$$

(where a, b, c, d are as in subsection 1.2.5).

Recalling the correspondence in (1.2.26) and Proposition 5.1.1, the matrix in the α, γ notation is :

$$\begin{pmatrix} \alpha^2 & -\mu(1 + \mu^{-2})^{\frac{1}{2}} \alpha \gamma^* & \mu^2 \gamma^{*2} \\ (1 + \mu^{-2})^{\frac{1}{2}} \alpha \gamma & I - \mu(\mu + \mu^{-1}) \gamma^* \gamma & -\mu(1 + \mu^{-2})^{\frac{1}{2}} \gamma^* \alpha^* \\ \gamma^2 & (1 + \mu^{-2})^{\frac{1}{2}} \gamma \alpha^* & \alpha^{*2} \end{pmatrix}.$$

Finally, in the symbols M, N, G, C, L , the above matrix is :

$$Z_1 = \begin{pmatrix} L & -\mu(1 + \mu^{-2})^{\frac{1}{2}} C & \mu^2 G^* \\ (1 + \mu^{-2})^{\frac{1}{2}} M & I - \mu(\mu + \mu^{-1}) N & -\mu(1 + \mu^{-2})^{\frac{1}{2}} M^* \\ G & (1 + \mu^{-2})^{\frac{1}{2}} C^* & L^* \end{pmatrix}. \quad (5.2.5)$$

As x_{-1}, x_0, x_1 generates $S_{\mu,c}^2$, the Woronowicz subalgebra of $SU_\mu(2)$ generated by the elements of the form $\langle \xi \otimes 1, \Delta_U(a)(\eta \otimes 1) \rangle$ is $SO_\mu(3)$ (where ξ, η are elements of \mathcal{H} , a belongs to $\mathcal{O}(S_{\mu,c}^2)$ and $\langle \cdot, \cdot \rangle$ is the $SU_\mu(2)$ -valued inner product of $\mathcal{H} \otimes SU_\mu(2)$).

5.3 Quantum Isometry Groups of the Podles sphere

Here we will compute $\widetilde{QISO}_R^+(S_{\mu,c}^2)$ with respect to the spectral triple given in [24] and show that it is isomorphic with $SO_\mu(3)$.

Let $(\tilde{\mathcal{Q}}, U)$ be an object in the category $\mathbf{Q}'_R(D)$ and \mathcal{Q} be the Woronowicz C^* subalgebra of $\tilde{\mathcal{Q}}$ generated by $\langle (\xi \otimes 1), \alpha_U(a)(\eta \otimes 1) \rangle_{\tilde{\mathcal{Q}}}$, for ξ, η belonging to \mathcal{H} , a belongs to $S_{\mu,c}^2$ (where $\langle \cdot, \cdot \rangle_{\tilde{\mathcal{Q}}}$ is the $\tilde{\mathcal{Q}}$ valued inner product of $\mathcal{H} \otimes \tilde{\mathcal{Q}}$). We shall

denote α_U by ϕ from now on.

The computation has the following steps:

Step 1. We prove that ϕ is ‘linear’, in the sense that it keeps the span of $\{1, A, B, B^*\}$ invariant.

Step 2. We shall exploit the facts that ϕ is a $*$ -homomorphism and preserves the canonical volume form on $S_{\mu,c}^2$, that is the restriction of the Haar state of $SU_\mu(2)$.

Step 3. We will compute the antipode of \mathcal{Q} and apply it to get some more relations.

Step 4. We will use the above steps to identify \mathcal{Q} as a subobject of $SO_\mu(3)$ which will finish the proof.

Remark 5.3.1. *The first step does not make use of the fact that α preserves the R -twisted volume, so linearity of the action follows for any object in the bigger category $\mathcal{Q}'(D)$.*

We now note down some useful facts for later use.

Lemma 5.3.2. *We observe :*

1. *The eigenspace of D corresponding to $(c_1 l + c_2)$ and $-(c_1 l + c_2)$ are $\text{span}\{v_{m, \frac{1}{2}}^l + v_{m, -\frac{1}{2}}^l : -1 \leq m \leq l\}$ and $\text{span}\{v_{m, \frac{1}{2}}^l - v_{m, -\frac{1}{2}}^l : -l \leq m \leq l\}$ respectively.*
2. *The eigenspace of $|D|$ corresponding to the eigenvalue $(c_1 \cdot \frac{1}{2} + c_2)$ is $\text{span}\{\alpha, \gamma, \alpha^*, \gamma^*\}$.*

Proof : 1. follows from (5.2.3). To prove 2., we note that by 1, it is sufficient to identify $\text{span}\{v_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}, v_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}, v_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}}, v_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}}\}$.

Using (5.2.1) and (1.2.27), we have:

$$\begin{aligned}
 v_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} &= N_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} F \triangleright (y_1^0 u_{\frac{1}{2}}) = N_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} F \triangleright (\alpha^* - s\gamma^*) \\
 &= N_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} (\gamma + \mu^{-1} s\alpha) = N_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} \gamma + \mu^{-1} s N_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} \alpha.
 \end{aligned}$$

$$\begin{aligned}
 v_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} &= N_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} F^0 \triangleright (y_1^0 u_{\frac{1}{2}}) = N_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} (\alpha^* - s\gamma^*) \\
 &= N_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} \alpha^* - s N_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}} \gamma^*.
 \end{aligned}$$

$$\begin{aligned}
v_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} &= N_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} F \triangleright (y_1^0 u_{-\frac{1}{2}}) = N_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} F \triangleright (E \triangleright w_{\frac{1}{2}}) \\
&= N_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} F \triangleright (E \triangleright (\alpha - \mu s \gamma)) = N_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} F \triangleright (-\mu \gamma^* - \mu s \alpha^*) \\
&= N_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} (\alpha - \mu s \gamma) = N_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} \alpha - \mu s N_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} \gamma.
\end{aligned}$$

$$\begin{aligned}
v_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} &= N_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} F^0 \triangleright (y_1^0 u_{-\frac{1}{2}}) = N_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} (E \triangleright (\alpha - \mu s \gamma)) \\
&= N_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} (-\mu \gamma^* - \mu s \alpha^*) = -\mu N_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} \gamma^* - \mu s N_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} \alpha^*.
\end{aligned}$$

Combining these, we have the result. \square

Lemma 5.3.3. 1. $\pi(A)v_{m,N}^l \in \text{span}\{v_{m,N}^{l-1}, v_{m,N}^l, v_{m,N}^{l+1}\},$

$$\pi(B)v_{m,N}^l \in \text{span}\{v_{m-1,N}^{l-1}, v_{m-1,N}^l, v_{m-1,N}^{l+1}\},$$

$$\pi(B^*)v_{m,N}^l \in \text{span}\{v_{m+1,N}^{l-1}, v_{m+1,N}^l, v_{m+1,N}^{l+1}\}.$$

$$2. \pi(A^k)(v_{m,N}^l) \in \text{span}\{v_{m,N}^{l-k}, v_{m,N}^{l-k+1}, \dots, v_{m,N}^{l+k}\}.$$

$$3. \pi(A^{m'} B^{n'})(v_{m,N}^l) \in \text{span}\{v_{m-n',N}^{l-m'-n'}, v_{m-n',N}^{l-(n'+m'-1)}, \dots, v_{m-n',N}^{l+n'+m'}\}.$$

$$4. \pi(A^r B^{*s})(v_{m,N}^l) \in \text{span}\{v_{m+s,N}^{l-s-r}, v_{m+s,N}^{l-s-r+1}, \dots, v_{m+s,N}^{l+s+r}\}.$$

Proof : Using (5.1.7) and (5.2.2), we have

$$\begin{aligned}
\pi(A)v_{m,N}^l &= \frac{1}{1+\mu^2} v_{m,N}^l - \frac{t^{-1}}{1+\mu^2} [\alpha_0^-(l, m; N) v_{m,N}^{l-1} + \alpha_0^0(l, m; N) v_{m,N}^l + \alpha_0^+(l, m; N) v_{m,N}^{l+1}].
\end{aligned}$$

Thus, $\pi(A)v_{m,N}^l$ belongs to $\text{span}\{v_{m,N}^{l-1}, v_{m,N}^l, v_{m,N}^{l+1}\}.$

Similarly, using the expressions for B and B^* from (5.1.7) and then using (5.2.2) just as above, we get the required statements for $\pi(B)$ and $\pi(B^*)$. This proves 1.

Repeated use of 1. now yields 2., 3. and 4. \square

5.3.1 Linearity of the action

For a vector v in \mathcal{H} , we shall denote by T_v the map from $\mathcal{B}(\mathcal{H})$ to $L^2(SU_\mu(2))$ given by $T_v(x) = xv \in \mathcal{H} \subset L^2(SU_\mu(2))$. It is clearly a continuous map w.r.t. the SOT on $\mathcal{B}(\mathcal{H})$

and the Hilbert space topology of $L^2(SU_\mu(2))$.

For an element a in $SU_\mu(2)$, we consider the right multiplication R_a as a bounded linear map on $L^2(SU_\mu(2))$. Clearly the composition $R_a T_v$ is a continuous linear map from $\mathcal{B}(\mathcal{H})$ (with SOT) to the Hilbert space $L^2(SU_\mu(2))$. We now define

$$T = R_{\alpha^*} T_\alpha + \mu^2 R_\gamma T_{\gamma^*}.$$

Lemma 5.3.4. *For any state ω on $\tilde{\mathcal{Q}}$ and x belonging to $S_{\mu,c}^2$, we have $T(\phi_\omega(x)) = \phi_\omega(x) \equiv R_1(\phi_\omega(x)) \in \overline{S_{\mu,c}^2} \subseteq L^2(SU_\mu(2))$, where $\phi_\omega(x) = (\text{id} \otimes \omega)(\phi(x))$.*

Proof : It is clear from the definition of T (using $\alpha\alpha^* + \mu^2\gamma\gamma^* = 1$) that $T(x) = x \equiv R_1(x)$ for x in $S_{\mu,c}^2 \subset \mathcal{B}(\mathcal{H})$, where x in the right hand side of the above denotes the identification of x in $S_{\mu,c}^2$ as a vector in $L^2(SU_\mu(2))$. Now, the lemma follows by noting that for x belonging to $S_{\mu,c}^2$, $\phi_\omega(x)$ belongs to $(S_{\mu,c}^2)''$, which is the SOT closure of $S_{\mu,c}^2$, and the SOT continuity of T discussed before. \square

Let

$$\mathcal{V}^l = \text{Span}\{v_{i,\pm\frac{1}{2}}^{l'}, -l' \leq i \leq l', l' \leq l\}.$$

As $\text{Span}\{v_{i,\pm\frac{1}{2}}^l, -l \leq i \leq l\}$, is the eigenspace of $|D|$ corresponding to the eigenvalue $c_1 l + c_2$, \tilde{U} and \tilde{U}^* keep \mathcal{V}^l invariant for all l .

Lemma 5.3.5. *There is some finite dimensional subspace \mathcal{V} of $\mathcal{O}(SU_\mu(2))$ such that $R_{\alpha^*}(\phi_\omega(A)v_{j,\pm\frac{1}{2}}^{\frac{1}{2}}), R_\gamma(\phi_\omega(A)v_{j,\pm\frac{1}{2}}^{\frac{1}{2}})$ belong to \mathcal{V} for all states ω on $\tilde{\mathcal{Q}}$.*

The same holds when A is replaced by B or B^ .*

Proof : We prove the result for A only, since a similar argument will work for B and B^* .

We have $\phi(A)(v_{j,\pm\frac{1}{2}}^{\frac{1}{2}} \otimes 1) = \tilde{U}(\pi(A) \otimes 1)\tilde{U}^*(v_{j,\pm\frac{1}{2}}^{\frac{1}{2}} \otimes 1)$.

Now, $\tilde{U}^*(v_{j,\pm\frac{1}{2}}^{\frac{1}{2}} \otimes 1)$ belong to $\mathcal{V}^{\frac{1}{2}} \otimes \tilde{\mathcal{Q}}$, and then using the definition of π as well as the Lemma 5.3.3, we get $(\pi(A) \otimes 1)\tilde{U}^*(v_{j,\pm\frac{1}{2}}^{\frac{1}{2}} \otimes 1)$ belong to $\text{Span}\{v_{j,\pm\frac{1}{2}}^{l'} : -l' \leq j \leq l', l' \leq \frac{3}{2}\} \otimes \tilde{\mathcal{Q}} = \mathcal{V}^{\frac{3}{2}} \otimes \tilde{\mathcal{Q}}$. Again, \tilde{U} keeps $\mathcal{V}^{\frac{3}{2}} \otimes \tilde{\mathcal{Q}}$ invariant, so $R_{\alpha^*}(\phi_\omega(A)(v_{j,\pm\frac{1}{2}}^{\frac{1}{2}}))$ belong to $\text{Span}\{v\alpha^*, v \in \mathcal{V}^{\frac{3}{2}}\}$. Similarly, $R_\gamma(\phi_\omega(A)(v_{j,\pm\frac{1}{2}}^{\frac{1}{2}}))$ belong to $\text{Span}\{v\gamma : v \in \mathcal{V}^{\frac{3}{2}}\}$. So, the lemma follows for A by taking $\mathcal{V} = \text{Span}\{v\alpha^*, v\gamma : v \in \mathcal{V}^{\frac{3}{2}}\} \subset \mathcal{O}(SU_\mu(2))$. \square

Since α, γ^* are in $\text{Span}\{v_{j,\pm\frac{1}{2}}^{\frac{1}{2}}\}$, we have the following immediate corollary:

Corollary 5.3.6. *There is a finite dimensional subspace \mathcal{V} of $\mathcal{O}(SU_\mu(2))$ such that for every state (hence for every bounded linear functional) ω on $\tilde{\mathcal{Q}}$, we have $T(\phi_\omega(A))$ belongs to \mathcal{V} . Similar conclusion holds for B and B^* as well.*

Proposition 5.3.7. $\phi(A), \phi(B), \phi(B^*)$ belong to $\mathcal{O}(S_{\mu,c}^2) \otimes_{\text{alg}} \mathcal{Q}$.

Proof : We give the proof for $\phi(A)$ only, the proof for B, B^* being similar. From the Corollary 5.3.6 and Lemma 5.3.4 it follows that for every bounded linear functional ω on $\tilde{\mathcal{Q}}$, we have $T(\phi_\omega(A))$ belongs to $\mathcal{V} \cap \overline{S_{\mu,c}^2} \subset \mathcal{O}(SU_\mu(2)) \cap \text{Ker}(\psi)$ (by Lemma 5.1.2) and hence $\mathcal{V} \cap \overline{S_{\mu,c}^2} = \mathcal{V} \cap \mathcal{O}(S_{\mu,c}^2)$, where \mathcal{V} is the finite dimensional subspace mentioned in Corollary 5.3.6. Clearly, $\mathcal{V} \cap \mathcal{O}(S_{\mu,c}^2)$ is a finite dimensional subspace of $\mathcal{O}(S_{\mu,c}^2)$ implying that there must be finite m , say, such that for every ω , $T(\phi_\omega(A))$ belongs to $\text{Span}\{A^k, A^k B^l, A^k B^{*l} : 0 \leq k, l \leq m\}$. Denote by \mathcal{W} the (finite dimensional) subspace of $\mathcal{B}(\mathcal{H})$ spanned by $\{A^k, A^k B^l, A^k B^{*l} : 0 \leq k, l \leq m\}$. Since for every state (and hence for every bounded linear functional) ω on $\tilde{\mathcal{Q}}$, we have $T(\phi_\omega(A)) = R_1(\phi_\omega(A)) \equiv \phi_\omega(A).1$, it is clear that $\phi_\omega(A)$ is in \mathcal{W} for every ω in $\tilde{\mathcal{Q}}^*$. Now, let us fix any faithful state ω on the separable unital C^* -algebra $\tilde{\mathcal{Q}}$ and embed $\tilde{\mathcal{Q}}$ in $\mathcal{B}(L^2(\mathcal{Q}, \omega)) \equiv \mathcal{B}(\mathcal{K})$. Thus, we get a canonical embedding of $\mathcal{L}(\mathcal{H} \otimes \tilde{\mathcal{Q}})$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Let us thus identify $\phi(A)$ as an element of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, and then by choosing a countable family of elements $\{q_1, q_2, \dots\}$ of $\tilde{\mathcal{Q}}$ which is an orthonormal basis in $\mathcal{K} = L^2(\omega)$, we can write $\phi(A)$ as a weakly convergent series of the form $\sum_{i,j=1}^\infty \phi^{ij}(A) \otimes |q_i\rangle\langle q_j|$. But $\phi^{ij}(A) = (\text{id} \otimes \omega_{ij})(\phi(A))$, where $\omega_{ij}(\cdot) = \omega(q_i^* \cdot q_j)$. So we have $\phi^{ij}(A)$ belongs to \mathcal{W} for all i, j , and hence the sequence $\sum_{i,j=1}^n \phi^{ij}(A) \otimes |q_i\rangle\langle q_j|$ belongs to $\mathcal{W} \otimes \mathcal{B}(\mathcal{K})$ converges weakly, and \mathcal{W} being finite dimensional (hence weakly closed), the limit, that is $\phi(A)$, must belong to $\mathcal{W} \otimes \mathcal{B}(\mathcal{K})$. In other words, if A_1, \dots, A_k denotes a basis of \mathcal{W} , we can write $\phi(A) = \sum_{i=1}^k A_i \otimes B_i$ for some B_i in $\mathcal{B}(\mathcal{K})$.

We claim that each B_i must belong to $\tilde{\mathcal{Q}}$. For any trace-class positive operator ρ in \mathcal{H} , say of the form $\rho = \sum_j \lambda_j |e_j\rangle\langle e_j|$, where $\{e_1, e_2, \dots\}$ is an orthonormal basis of \mathcal{H} and $\lambda_j \geq 0, \sum_j \lambda_j < \infty$, let us denote by ψ_ρ the normal functional on $\mathcal{B}(\mathcal{H})$ given by $x \mapsto \text{Tr}(\rho x)$, and it is easy to see that it has a canonical extension $\tilde{\psi}_\rho := (\psi_\rho \otimes \text{id})$ on $\mathcal{L}(\mathcal{H} \otimes \tilde{\mathcal{Q}})$ given by $\tilde{\psi}_\rho(X) = \sum_j \lambda_j \langle e_j \otimes 1, X(e_j \otimes 1) \rangle_{\tilde{\mathcal{Q}}}$, where X is in $\mathcal{L}(\mathcal{H} \otimes \tilde{\mathcal{Q}})$ and $\langle \cdot, \cdot \rangle_{\tilde{\mathcal{Q}}}$ denotes that $\tilde{\mathcal{Q}}$ valued inner product of $\mathcal{H} \otimes \tilde{\mathcal{Q}}$. Clearly, $\tilde{\psi}_\rho$ is a bounded linear map from $\mathcal{L}(\mathcal{H} \otimes \tilde{\mathcal{Q}})$ to $\tilde{\mathcal{Q}}$. Now, since A_1, \dots, A_k in the expression of $\phi(A)$ are linearly independent, we can choose trace class operators ρ_1, \dots, ρ_n such that $\psi_{\rho_i}(A_i) = 1$ and $\psi_{\rho_i}(A_j) = 0$ for $j \neq i$. Then, by applying $\tilde{\psi}_{\rho_i}$ on $\phi(A)$ we conclude that B_i belongs to $\tilde{\mathcal{Q}}$. But by definition of \mathcal{Q} as the Woronowicz subalgebra of $\tilde{\mathcal{Q}}$ generated by $\langle \xi \otimes 1, \phi(x)(\eta \otimes 1) \rangle_{\tilde{\mathcal{Q}}}$, with η, ξ belonging to \mathcal{H} , we must have B_i belongs to \mathcal{Q} . \square

Proposition 5.3.8. ϕ keeps the span of $1, A, B, B^*$ invariant.

Proof : We prove the result for $\phi(A)$. The proof for the others are exactly similar.

Using Proposition 5.3.7, we can write $\phi(A)$ as a finite sum of the form $\sum_{k \geq 0} A^k \otimes Q_k + \sum_{m', n', n' \neq 0} A^{m'} B^{n'} \otimes R_{m', n'} + \sum_{r, s, s \neq 0} A^r B^{*s} \otimes R'_{r, s}$.

Let $\xi = v_{m_0, N_0}^l$.

We have $U(\xi)$ belongs to $\text{Span}\{v_{m,N}^l, m = -l, \dots, l, N = \pm \frac{1}{2}\}$. Let us write

$$\tilde{U}(\xi \otimes 1) = \sum_{m=-l, \dots, l, N=\pm \frac{1}{2}} v_{m,N}^l \otimes q_{(m,N), (m_0, N_0)}^l,$$

where $q_{(m,N), (m_0, N_0)}^l$ belong to \mathcal{Q} . Since α_U preserves the R -twisted volume, we have :

$$\sum_{m', N'} q_{(m, N), (m', N')}^{l*} q_{(m, N), (m', N')}^{l*} = 1. \quad (5.3.1)$$

It also follows that $U(A\xi)$ belong to $\text{Span}\{v_{m, N}^{l'}, m = -l', \dots, l', l' = l-1, l, l+1, N = \pm \frac{1}{2}\}$.

Recalling Lemma 5.3.3, we have $\phi(A)\tilde{U}(\xi \otimes 1) = \sum_{k, m=-l, \dots, l, N=\pm \frac{1}{2}} A^k v_{m, N}^l \otimes Q_k q_{(m, N), (m_0, N_0)}^l + \sum_{m', n', n' \neq 0, m=-l, \dots, l, N=\pm \frac{1}{2}} A^{m'} B^{n'} v_{m, N}^l \otimes R_{m', n'} q_{(m, N), (m_0, N_0)}^l + \sum_{r, s, s \neq 0, m=-l, \dots, l, N=\pm \frac{1}{2}} A^r B^{*s} v_{m, N}^l \otimes R'_{r, s} q_{(m, N), (m_0, N_0)}^l$.

Let m'_0 denote the largest integer m' such that there is a nonzero coefficient of $A^{m'} B^{n'}, n' \geq 1$ in the expression of $\phi(A)$. We claim that the coefficient of $v_{m-n', N}^{l-m'_0-n'}$ in $\phi(A)\tilde{U}(\xi \otimes 1)$ is $R_{m'_0, n'} q_{(m, N), (m_0, N_0)}^l$.

Indeed, the term $v_{m-n', N}^{l-m'_0-n'}$ can arise in three ways: it can come from a term of the form $A^{m''} B^{n''} v_{m, N}^l$ or $A^k v_{m, N}^l$ or $A^r B^{*s} v_{m, N}^l$ for some m'', n'', k, r, s .

In the first case, using Lemma 5.3.3, we must have $l-m'_0-n' = l-m''-n''+t, 0 \leq t \leq 2m''$ and $m-n' = m-n''$ implying $m'' = m'_0+t$, and since m'_0 is the largest integer such that $A^{m'_0} B^{n'}$ appears in $\phi(A)$, we only have the possibility $t=0$, that is $v_{m-n', N}^{l-m'_0-n'}$ appears only in $A^{m'_0} B^{n'}$.

In the second case, we have $m-n' = m$ implying $n' = 0$ - a contradiction. In the last case, we have $m-n' = m+s$ so that $-n' = s$ which is only possible when $n' = s = 0$ which is again a contradiction.

Now, coefficient of $v_{m-n', N}^{l-m'_0-n'}$ in $\tilde{U}(A\xi \otimes 1)$ is zero if $m'_0 \geq 1$ (as $n' \neq 0$). It now follows from the above claim, using Lemma 5.3.3 and comparing coefficients in the equality $\tilde{U}(A\xi \otimes 1) = \phi(A)\tilde{U}(\xi \otimes 1)$, that $R_{m'_0, n'} q_{(m, N), (m_0, N_0)}^l = 0$ for all $n' \geq 1$, for all m, N when $m'_0 \geq 1$. Now varying (m_0, N_0) , we conclude that the above holds for all (m_0, N_0) . Using (5.3.1), we conclude that

$$R_{m'_0, n'} \sum_{m', N'} q_{(m, N), (m', N')}^{l*} q_{(m, N), (m', N')}^{l*} = 0 \text{ for all } n' \geq 1,$$

that is, $R_{m'_0, n'} = 0$ for all $n' \geq 1$ if $m'_0 \geq 1$. Proceeding by induction on m'_0 , we deduce $R_{m', n'} = 0$ for all $m' \geq 1, n' \geq 1$.

Similarly, we have $Q_k = 0$ for all $k \geq 2$ and $R'_{r, s} = 0$ for all $r \geq 1, s \geq 1$.

Thus, $\phi(A)$ belongs to $\text{span}\{1, A, B, B^*, B^2, \dots, B^n, B^{*2}, \dots, B^{*m}\}$. But the coefficient of $v_{m-n', N}^{l-n'}$ in $\phi(A)\tilde{U}(\xi \otimes 1)$ is $R_{0, n'}$. Arguing as before, we conclude that $R_{0, n'} = 0$ for all $n' \geq 2$. In a similar way, we can prove $R'_{0, n'} = 0$ for all $n' \geq 2$. \square

In view of the above, let us write:

$$\phi(A) = 1 \otimes T_1 + A \otimes T_2 + B \otimes T_3 + B^* \otimes T_4, \quad (5.3.2)$$

$$\phi(B) = 1 \otimes S_1 + A \otimes S_2 + B \otimes S_3 + B^* \otimes S_4, \quad (5.3.3)$$

for some T_i, S_i in \mathcal{Q} .

5.3.2 Homomorphism conditions

In this subsection, we shall use the facts that ϕ is a $*$ -homomorphism and it preserves the R -twisted volume to derive relations among T_i, S_i in (5.3.2), (5.3.3).

Lemma 5.3.9.

$$T_1 = \frac{1 - T_2}{1 + \mu^2},$$

$$S_1 = \frac{-S_2}{1 + \mu^2}.$$

Proof : We have the expressions of A and B in terms of the $SU_\mu(2)$ elements from the equations (5.1.17), (5.1.18) and (5.1.19). From these, we note that $h(A) = (1 + \mu^2)^{-1}$ and $h(B) = 0$. By recalling Proposition 5.2.2, we use $(h \otimes \text{id})\phi(A) = h(A).1$ and $(h \otimes \text{id})\phi(B) = h(B).1$ to have the above two equations. \square

Lemma 5.3.10. $T_1^* = T_1, T_2^* = T_2, T_4^* = T_3$.

Proof : Follows by comparing the coefficients of $1, A$ and B respectively in the equation $\phi(A^*) = \phi(A)$. \square

We shall now assume that $\mu \neq 1$. The case $\mu = 1$ will be discussed separately.

Lemma 5.3.11.

$$S_2^* S_2 + c(1 + \mu^2)^2 S_3^* S_3 + c(1 + \mu^2)^2 S_4^* S_4$$

$$= (1 - T_2)(\mu^2 + T_2) - c(1 + \mu^2)^2 T_3 T_3^* - c(1 + \mu^2)^2 T_3^* T_3 + c(1 + \mu^2)^2 .1, \quad (5.3.4)$$

$$-2S_2^* S_2 + (1 + \mu^2) S_3^* S_3 + \mu^2 (1 + \mu^2) S_4^* S_4 = (\mu^2 + 2T_2 - 1)T_2 - \mu^2 (1 + \mu^2) T_3 T_3^* - (1 + \mu^2) T_3^* T_3, \quad (5.3.5)$$

$$S_2^* S_2 - S_3^* S_3 - \mu^4 S_4^* S_4 = -T_2^2 + \mu^4 T_3 T_3^* + T_3^* T_3, \quad (5.3.6)$$

$$S_2^* S_4 + S_3^* S_2 = -(\mu^2 + T_2)T_3^* + T_3^* (1 - T_2), \quad (5.3.7)$$

$$S_2^* S_3 + \mu^2 S_4^* S_2 = -T_2 T_3 - \mu^2 T_3 T_2, \quad (5.3.8)$$

$$S_4^* S_3 = -T_3^2. \quad (5.3.9)$$

Proof : It follows by comparing the coefficients of $1, A, A^2, B^*, AB$ and B^2 in the equation $\phi(B^*B) = \phi(A) - \phi(A^2) + cI$ and then substituting S_1, T_1, T_2^*, T_4 by $\frac{-S_2}{1+\mu^2}, \frac{1-T_2}{1+\mu^2}, T_2, T_3^*$ respectively by using the relations in Lemma 5.3.9 and Lemma 5.3.10. \square

Lemma 5.3.12.

$$\begin{aligned} & -S_2(1 - T_2) + c(1 + \mu^2)^2 S_3 T_3^* + c(1 + \mu^2)^2 S_4 T_3 \\ & = -\mu^2(1 - T_2)S_2 + c\mu^2(1 + \mu^2)^2 T_3 S_4 + c\mu^2(1 + \mu^2)^2 T_3^* S_3, \end{aligned} \quad (5.3.10)$$

$$S_2 - 2S_2 T_2 + (1 + \mu^2)(\mu^2 S_3 T_3^* + S_4 T_3) = \mu^2 S_2 - 2\mu^2 T_2 S_2 + \mu^4(1 + \mu^2) T_3 S_4 + \mu^2(1 + \mu^2) T_3^* S_3, \quad (5.3.11)$$

$$-S_2 T_3 + S_3(1 - T_2) = -\mu^2 T_3 S_2 + \mu^2(1 - T_2) S_3, \quad (5.3.12)$$

$$-S_2 T_3^* + S_4(1 - T_2) = \mu^2(1 - T_2) S_4 - \mu^2 T_3^* S_2, \quad (5.3.13)$$

$$S_2 T_3 + \mu^2 S_3 T_2 = \mu^2(T_2 S_3 + \mu^2 T_3 S_2), \quad (5.3.14)$$

$$S_3 T_3 = \mu^2 T_3 S_3, \quad (5.3.15)$$

$$S_4 T_3^* = \mu^2 T_3^* S_4. \quad (5.3.16)$$

Proof : It follows by equating the coefficients of $1, A, B, B^*, AB, B^2$ and B^{*2} in the equation $\phi(BA) = \mu^2 \phi(AB)$ and then using Lemma 5.3.9 and Lemma 5.3.10. \square

Lemma 5.3.13.

$$\begin{aligned} & S_2 S_2^* + c(1 + \mu^2)^2 S_3 S_3^* + c(1 + \mu^2)^2 S_4 S_4^* \\ & = \mu^2(1 - T_2)(1 + \mu^2 T_2) + c(1 + \mu^2)^2 T_3 T_3^* + c(1 + \mu^2)^2 T_3^* T_3 + c(1 + \mu^2)^2 \cdot 1, \end{aligned} \quad (5.3.17)$$

$$\begin{aligned} & -2S_2 S_2^* + \mu^2(1 + \mu^2) S_3 S_3^* + (1 + \mu^2) S_4 S_4^* \\ & = \mu^2(1 + \mu^2) T_2 - 2\mu^4(1 - T_2) T_2 - \mu^6(1 + \mu^2) T_3 T_3^* - \mu^4(1 + \mu^2) T_3^* T_3, \end{aligned} \quad (5.3.18)$$

$$-S_2 S_4^* - S_3 S_2^* = \mu^2(1 + \mu^2) T_3 - \mu^4(1 - T_2) T_3 - \mu^4 T_3(1 - T_2), \quad (5.3.19)$$

$$S_2 S_2^* - \mu^4 S_3 S_3^* - S_4 S_4^* = -\mu^4 T_2^2 + \mu^8 T_3 T_3^* + \mu^4 T_3^* T_3, \quad (5.3.20)$$

$$S_2 S_4^* + \mu^2 S_3 S_2^* = -\mu^4 T_2 T_3 - \mu^6 T_3 T_2, \quad (5.3.21)$$

$$S_3 S_4^* = -\mu^4 T_3^2. \quad (5.3.22)$$

Proof : The Lemma is proved by equating the coefficient of $1, A, B, A^2, AB, B^2$

in the equation $\phi(BB^*) = \mu^2\phi(A) - \mu^4\phi(A^2) + c.1$ and then using Lemma 5.3.9 and Lemma 5.3.10. \square

5.3.3 Relations from the antipode

Now, we compute the antipode, say κ of $\tilde{\mathcal{Q}}$.

To begin with, we recall from Lemma 5.1.3 and Proposition 5.1.5 that $\{x_{-1}, x_0, x_1\}$ is a set of orthogonal vectors with same norm.

Lemma 5.3.14. *If x'_{-1}, x'_0, x'_1 is the normalized basis corresponding to $\{x_{-1}, x_0, x_1\}$, then from (5.3.2) and (5.3.3) we obtain*

$$\begin{aligned}\phi(x'_{-1}) &= x'_{-1} \otimes S_3 + x'_0 \otimes -\mu^{-1}(1 + \mu^2)^{-\frac{1}{2}}S_2 + x'_1 \otimes -\mu^{-1}S_4, \\ \phi(x'_0) &= x'_{-1} \otimes -\mu(1 + \mu^2)^{\frac{1}{2}}T_3 + x'_0 \otimes T_2 + x'_1 \otimes (1 + \mu^2)^{\frac{1}{2}}T_4, \\ \phi(x'_1) &= x'_{-1} \otimes -\mu S_4^* + x'_0 \otimes (1 + \mu^2)^{-\frac{1}{2}}S_2^* + x'_1 \otimes S_3^*.\end{aligned}$$

Proof : As x_{-1}, x_0, x_1 have same norm, it follows that $x'_i = Kx_i$, where $K = \|x_i\|^{-1}$, $i = \{-1, 0, 1\}$.

Now, using (5.1.7) and (5.3.3), we have

$$\begin{aligned}\phi(x'_{-1}) &= \frac{Kt(1 + \mu^2)^{\frac{1}{2}}}{\mu}\phi(B) \\ &= \frac{Kt(1 + \mu^2)^{\frac{1}{2}}}{\mu}\left[1 \otimes S_1 + \frac{1 - t^{-1}x_0}{1 + \mu^2} \otimes S_2 + \frac{\mu x_{-1}}{t(1 + \mu^2)^{\frac{1}{2}}} \otimes S_3 + \frac{\mu(-\mu^{-1}x_1)}{t(1 + \mu^2)^{\frac{1}{2}}} \otimes S_4\right] \\ &= Kx_{-1} \otimes S_3 + Kx_0 \otimes -\frac{S_2}{\mu(1 + \mu^2)^{\frac{1}{2}}} + Kx_1 \otimes -\frac{S_4}{\mu}\end{aligned}$$

(by Lemma 5.3.9)

$$= x'_{-1} \otimes S_3 + x'_0 \otimes -\mu^{-1}(1 + \mu^2)^{-\frac{1}{2}}S_2 + x'_1 \otimes -\mu^{-1}S_4.$$

By similar calculations, we get the second and the third equations. \square

Hence, ϕ keeps the span of the orthonormal set $\{x'_{-1}, x'_0, x'_1\}$ invariant. Moreover, ϕ is kept invariant by the Haar state h of $SU_\mu(2)$. Therefore, we have a unitary representation of the CQG $\tilde{\mathcal{Q}}$ on span $\{x'_{-1}, x'_0, x'_1\}$.

Using $T_4 = T_3^*$ from Lemma 5.3.10, the unitary matrix, say Z corresponding to ϕ and the ordered basis $\{x'_{-1}, x'_0, x'_1\}$ is given by :

$$Z = \begin{pmatrix} S_3 & -\mu\sqrt{1+\mu^2}T_3 & -\mu S_4^* \\ \frac{-S_2}{\mu\sqrt{1+\mu^2}} & T_2 & \frac{S_2^*}{\sqrt{1+\mu^2}} \\ -\mu^{-1}S_4 & \sqrt{1+\mu^2}T_3^* & S_3^* \end{pmatrix}. \quad (5.3.23)$$

Recall that (cf [41]), the antipode κ on the matrix elements of a finite-dimensional unitary representation $U^\alpha \equiv (u_{pq}^\alpha)$ is given by $\kappa(u_{pq}^\alpha) = (u_{qp}^\alpha)^*$. Hence, the antipode is given by :

$$\kappa(T_2) = T_2, \quad \kappa(T_3) = \frac{S_2^*}{\mu^2(1+\mu^2)}, \quad \kappa(S_2) = \mu^2(1+\mu^2)T_3^*,$$

$$\kappa(S_3) = S_3^*, \quad \kappa(S_4) = \mu^2 S_4, \quad \kappa(T_3^*) = \frac{S_2}{1+\mu^2},$$

$$\kappa(S_2^*) = (1+\mu^2)T_3, \quad \kappa(S_3^*) = S_3, \quad \kappa(S_4^*) = \mu^{-2}S_4^*.$$

Now we derive some more equations by applying κ on the equations obtained by homomorphism condition.

Lemma 5.3.15.

$$\begin{aligned} & \mu^4(1+\mu^2)^2 T_3^* T_3 + c\mu^2(1+\mu^2)^2 S_3^* S_3 + c\mu^2(1+\mu^2)^2 S_4 S_4^* \\ &= \mu^2(1-T_2)(\mu^2+T_2) - cS_2 S_2^* - cS_2^* S_2 + c\mu^2(1+\mu^2)^2 \cdot 1, \end{aligned} \quad (5.3.24)$$

$$\begin{aligned} & -2\mu^4(1+\mu^2)^3 T_3^* T_3 + \mu^2(1+\mu^2)^2 S_3^* S_3 + \mu^4(1+\mu^2)^2 S_4 S_4^* \\ &= \mu^2(1+\mu^2)T_2(\mu^2+2T_2-1) - \mu^2 S_2 S_2^* - S_2^* S_2, \end{aligned} \quad (5.3.25)$$

$$\begin{aligned} & \mu^4(1+\mu^2)^4 T_3^* T_3 - \mu^2(1+\mu^2)^2 S_3^* S_3 - \mu^6(1+\mu^2)^2 S_4 S_4^* \\ &= -\mu^2(1+\mu^2)^2 T_2^2 + \mu^4 S_2 S_2^* + S_2^* S_2, \end{aligned} \quad (5.3.26)$$

$$\mu^2(1+\mu^2)^2 S_4 T_3 + \mu^2(1+\mu^2)^2 T_3^* S_3 = -S_2(\mu^2+T_2) + (1-T_2)S_2, \quad (5.3.27)$$

$$S_4 S_3 = -\frac{-S_2^2}{\mu^2(1+\mu^2)^2}. \quad (5.3.28)$$

Proof : The relations follow by applying κ on (5.3.4), (5.3.5), (5.3.6), (5.3.7) and (5.3.9) respectively. \square

Lemma 5.3.16.

$$-\mu^2(1-T_2)T_3^* + cS_2S_3^* + cS_2^*S_4 = -\mu^4T_3^*(1-T_2) + c\mu^2S_4S_2^* + c\mu^2S_3^*S_2, \quad (5.3.29)$$

$$S_3S_2 = \mu^2S_2S_3, \quad (5.3.30)$$

$$S_2S_4 = \mu^2S_4S_2, \quad (5.3.31)$$

$$-S_2^*T_3^* + (1-T_2)S_3^* = -\mu^2T_3^*S_2^* + \mu^2S_3^*(1-T_2), \quad (5.3.32)$$

$$-S_2T_3^* + (1-T_2)S_4 = \mu^2S_4(1-T_2) - \mu^2T_3^*S_2, \quad (5.3.33)$$

$$T_3S_2 + \mu^2S_3T_2 = \mu^2(T_2S_3 + \mu^2S_2T_3). \quad (5.3.34)$$

Proof : The relations follow by applying κ on (5.3.10), (5.3.15), (5.3.16), (5.3.12), (5.3.13) and (5.3.14) respectively. \square

Lemma 5.3.17.

$$\begin{aligned} & \mu^4(1+\mu^2)^2T_3T_3^* + c\mu^2(1+\mu^2)^2S_3S_3^* + c\mu^2(1+\mu^2)^2S_4^*S_4 \\ &= \mu^4(1-T_2)(1+\mu^2T_2) + cS_2S_2^* + cS_2^*S_2 + c\mu^2(1+\mu^2)^2.1, \end{aligned} \quad (5.3.35)$$

$$S_3S_4 = -\frac{\mu^2S_2^2}{(1+\mu^2)^2}, \quad (5.3.36)$$

$$-\mu^2(1+\mu^2)^2S_4^*T_3^* - \mu^2(1+\mu^2)^2T_3S_3^* = \mu^2(1+\mu^2)S_2^* - \mu^4S_2^*(1-T_2) - \mu^4(1-T_2)S_2^*, \quad (5.3.37)$$

$$(1+\mu^2)^2S_4^*T_3^* + \mu^2(1+\mu^2)^2T_3S_3^* = -\mu^2S_2^*T_2 - \mu^4T_2S_2^*. \quad (5.3.38)$$

Proof : The relations follow by applying κ on (5.3.17), (5.3.22), (5.3.19) and (5.3.21) respectively. \square

Remark 5.3.18. It follows from (5.3.28) and (5.3.36) that $\mu^4S_4S_3 = S_3S_4$.

5.3.4 Identification of $SO_\mu(3)$ as the quantum isometry group

Motivated by (5.2.5) and (5.3.23) we are led to state and prove the following statement :

The map $SO_\mu(3) \rightarrow \mathcal{Q}$ sending M, L, G, N, C to $-(1+\mu^2)^{-1}S_2, S_3, -\mu^{-1}S_4, (1+\mu^2)^{-1}(1-T_2), \mu T_3$ respectively is a $*$ homomorphism (See Proposition 5.3.30).

To prove this, it is enough to show that all the relations of $SO_\mu(3)$ (as in (5.2.4)) when translated to relations of $QISO_R^+(\mathcal{O}(S_{\mu,c}^2), \mathcal{H}, D)$ via the above map are satisfied. Hence, we list the relations one by one.

$$L^*L = (I - N)(I - \mu^{-2}N) \text{ gives}$$

$$S_3^* S_3 = (1 - \frac{1-T_2}{1+\mu^2})(1 - \mu^{-2} \frac{1-T_2}{1+\mu^2}) = \frac{\mu^2+T_2}{1+\mu^2} (\frac{\mu^2(1+\mu^2)-(1-T_2)}{\mu^2(1+\mu^2)}),$$

which implies

$$\mu^2(1+\mu^2)^2 S_3^* S_3 = (\mu^2 + T_2)(\mu^2(1+\mu^2) - (1-T_2)).$$

$$LL^* = (1 - \mu^2 N)(1 - \mu^4 N) \text{ gives } S_3 S_3^* = (1 - \mu^2(\frac{1-T_2}{1+\mu^2}))(1 - \mu^4(\frac{1-T_2}{1+\mu^2})) \text{ implying}$$

$$(1 + \mu^2)^2 S_3 S_3^* = (1 + \mu^2 T_2)((1 + \mu^2) - \mu^4(1 - T_2)).$$

$$G^* G = G G^* = N^2 \text{ gives } -\frac{S_4^*}{\mu}(-\frac{S_4}{\mu}) = (-\frac{S_4}{\mu})(-\frac{S_4^*}{\mu}) = \frac{(1-T_2)^2}{(1+\mu^2)^2} \text{ implying}$$

$$S_4^* S_4 = S_4 S_4^* = \frac{\mu^2(1-T_2)^2}{(1+\mu^2)^2}.$$

$$M^* M = N - N^2 \text{ gives}$$

$$\frac{S_2^* S_2}{(1+\mu^2)^2} = \frac{1-T_2}{1+\mu^2} - \frac{(1-T_2)^2}{(1+\mu^2)^2},$$

$$\text{which means } \frac{S_2^* S_2}{(1+\mu^2)^2} = \frac{(1+\mu^2)(1-T_2)-(1-T_2)^2}{(1+\mu^2)^2} \text{ implying}$$

$$S_2^* S_2 = (1-T_2)(\mu^2 + T_2).$$

$$MM^* = \mu^2 N - \mu^4 N^2 \text{ gives } \frac{S_2 S_2^*}{(1+\mu^2)^2} = \mu^2(\frac{1-T_2}{1+\mu^2}) - \mu^4(\frac{1-T_2}{1+\mu^2})^2, \text{ which implies}$$

$$S_2 S_2^* = \mu^2(1-T_2)(1+\mu^2 T_2).$$

$$C^* C = N - N^2 \text{ gives } \mu T_3^* \mu T_3 = \frac{1-T_2}{1+\mu^2} - (\frac{1-T_2}{1+\mu^2})^2, \text{ which implies}$$

$$(1-T_2)(\mu^2 + T_2) = \mu^2(1+\mu^2)^2 T_3^* T_3.$$

$$CC^* = \mu^2 N - \mu^4 N^2 \text{ gives } (\mu T_3)(\mu T_3)^* = \mu^2(\frac{1-T_2}{1+\mu^2}) - \mu^4(\frac{1-T_2}{1+\mu^2})^2, \text{ which implies } \mu^2 T_3 T_3^* = \frac{\mu^2(1+\mu^2)(1-T_2)-\mu^4(1-T_2)^2}{(1+\mu^2)^2}. \text{ Therefore, } (1+\mu^2)(1-T_2)-\mu^2(1-T_2)^2 = (1+\mu^2)^2 T_3 T_3^*. \text{ Thus,}$$

$$(1-T_2)(1+\mu^2 T_2) = (1+\mu^2)^2 T_3 T_3^*.$$

$$LN = \mu^4 NL \text{ gives } S_3(\frac{1-T_2}{1+\mu^2}) = \mu^4(\frac{1-T_2}{1+\mu^2}) S_3, \text{ implying}$$

$$S_3(1-T_2) = \mu^4(1-T_2) S_3.$$

$$GN = NG \text{ gives } -\frac{S_4}{\mu}(\frac{1-T_2}{1+\mu^2}) = (\frac{1-T_2}{1+\mu^2})(-\frac{S_4}{\mu}), \text{ which implies } S_4(1-T_2) = (1-T_2) S_4.$$

Thus,

$$S_4 T_2 = T_2 S_4.$$

$MN = \mu^2 NM$ gives $(-\frac{S_2}{1+\mu^2})(\frac{1-T_2}{1+\mu^2}) = \mu^2(\frac{1-T_2}{1+\mu^2})(-\frac{S_2}{1+\mu^2})$, implying

$$S_2(1-T_2) = \mu^2(1-T_2)S_2.$$

$CN = \mu^2 NC$ gives $\mu T_3(\frac{1-T_2}{1+\mu^2}) = \mu^2(\frac{1-T_2}{1+\mu^2})\mu T_3$, which implies

$$T_3(1-T_2) = \mu^2(1-T_2)T_3.$$

$LG = \mu^4 GL$ gives $S_3(-\mu^{-1}S_4) = \mu^4(-\mu^{-1}S_4)S_3$, that is,

$$S_3 S_4 = \mu^4 S_4 S_3.$$

$LM = \mu^2 ML$ gives $S_3(-\frac{S_2}{1+\mu^2}) = \mu^2(-\frac{S_2}{1+\mu^2})S_3$, that is,

$$S_3 S_2 = \mu^2 S_2 S_3.$$

$MG = \mu^2 GM$ gives $(-\frac{S_2}{1+\mu^2})(-\frac{S_4}{\mu}) = \mu^2(-\frac{S_4}{\mu})(-\frac{S_2}{1+\mu^2})$, that is,

$$S_2 S_4 = \mu^2 S_4 S_2.$$

$CM = MC$ gives $(\mu T_3)(-\frac{S_2}{1+\mu^2}) = (-\frac{S_2}{1+\mu^2})(\mu T_3)$, that is,

$$T_3 S_2 = S_2 T_3.$$

$LG^* = \mu^4 G^* L$ gives $S_3(-\frac{S_4^*}{\mu}) = \mu^4(-\frac{S_4^*}{\mu})S_3$, that is,

$$S_3 S_4^* = \mu^4 S_4^* S_3.$$

$M^2 = \mu^{-1} LG$ gives $(-\frac{S_2}{1+\mu^2})^2 = \mu^{-1} S_3(-\frac{S_4}{\mu})$, that is,

$$S_3 S_4 = -\frac{\mu^2}{(1+\mu^2)^2} S_2^2.$$

$M^* L = \mu^{-1}(I-N)C$ gives $-\frac{S_2^*}{1+\mu^2} S_3 = \mu^{-1}(1-\frac{1-T_2}{1+\mu^2})\mu T_3$, that is,

$$-S_2^* S_3 = (\mu^2 + T_2) T_3.$$

$N^* = N$ gives $\frac{(1-T_2)^*}{1+\mu^2} = \frac{1-T_2}{1+\mu^2}$, that is,

$$T_2^* = T_2.$$

Thus, we are led to prove the following lemmas, in some of which we will need $\mu^2 \neq 1$. The case $\mu = 1$ will be dealt separately.

Lemma 5.3.19. $S_2^* S_2 = (1 - T_2)(\mu^2 + T_2)$.

Proof : Subtracting the equation obtained by multiplying $c(1 + \mu^2)$ with (5.3.5) from (5.3.4), we have

$$\begin{aligned} & (1 + 2c(1 + \mu^2))S_2^* S_2 + c(1 + \mu^2)^2(1 - \mu^2)S_4^* S_4 \\ &= (1 - T_2)(\mu^2 + T_2) - c(1 + \mu^2)(\mu^2 + 2T_2 - 1)T_2 + c(1 + \mu^2)^2(\mu^2 - 1)T_3 T_3^* + c(1 + \mu^2)^2.1. \end{aligned} \quad (5.3.39)$$

Again, by adding (5.3.4) with $c(1 + \mu^2)^2$ times (5.3.6) gives

$$\begin{aligned} & (1 + c(1 + \mu^2)^2)S_2^* S_2 + c(1 - \mu^4)(1 + \mu^2)^2 S_4^* S_4 \\ &= (1 - T_2)(\mu^2 + T_2) - c(1 + \mu^2)^2 T_2^2 + c(1 + \mu^2)^2(\mu^4 - 1)T_3 T_3^* + c(1 + \mu^2)^2.1. \end{aligned} \quad (5.3.40)$$

Subtracting the equation obtained by multiplying $(\mu^2 + 1)$ with (5.3.39) from (5.3.40) we obtain

$$\begin{aligned} & -(\mu^2 + c(1 + \mu^2)^2)S_2^* S_2 = (1 - T_2)(\mu^2 + T_2) - c(1 + \mu^2)^2 T_2^2 \\ & -(1 + \mu^2)(1 - T_2)(\mu^2 + T_2) - c\mu^2(1 + \mu^2)^2.1 + c(1 + \mu^2)^2(\mu^2 + 2T_2 - 1)T_2. \end{aligned}$$

The right hand side can be seen to equal $-(\mu^2 + c(1 + \mu^2)^2)(1 - T_2)(\mu^2 + T_2)$. Thus, $S_2^* S_2 = (1 - T_2)(\mu^2 + T_2)$. \square

Lemma 5.3.20.

$$\mu^2(1 + \mu^2)^2 T_3^* T_3 = (1 - T_2)(\mu^2 + T_2), \quad (5.3.41)$$

$$(1 + \mu^2)^2 T_3 T_3^* = (1 - T_2)(1 + \mu^2 T_2), \quad (5.3.42)$$

$$S_2 S_2^* = \mu^2(1 - T_2)(1 + \mu^2 T_2). \quad (5.3.43)$$

Proof : Applying κ on Lemma 5.3.19, we obtain (5.3.41).

Unitarity of the matrix Z ((2, 2) position of the matrix $Z^* Z$) gives $\mu^2(1 + \mu^2)T_3^* T_3 + T_2^2 + (1 + \mu^2)T_3 T_3^* = 1$.

Using (5.3.41) we deduce $-(1 + \mu^2)^2 T_3 T_3^* = (T_2 - 1)(1 + \mu^2 T_2)$. Thus we obtain (5.3.42).

Applying κ on (5.3.42), we deduce (5.3.43). \square

Lemma 5.3.21. $S_4^* S_4 = S_4 S_4^* = (1 + \mu^2)^{-2} \mu^2 (1 - T_2)^2$.

Proof : Adding (5.3.25) and (5.3.26), we have : $-\mu^4(1 + \mu^2)^3(1 - \mu^2)T_3^*T_3 + \mu^4(1 + \mu^2)^2(1 - \mu^2)S_4S_4^* = -\mu^2(1 + \mu^2)(1 - \mu^2)T_2(1 - T_2) - \mu^2(1 - \mu^2)S_2S_2^*$.

Using $\mu^2 \neq 1$, we obtain,

$$-\mu^4(1 + \mu^2)^3 T_3^* T_3 + \mu^4(1 + \mu^2)^2 S_4 S_4^* = -\mu^2(1 + \mu^2) T_2 (1 - T_2) - \mu^2 S_2 S_2^*.$$

Now using (5.3.41) and (5.3.43), we reduce the above equation to

$$\begin{aligned} & \mu^4(1 + \mu^2)^2 S_4 S_4^* \\ &= -\mu^2(1 - T_2)(T_2 + \mu^2 T_2 + \mu^2 + \mu^4 T_2) + \mu^2(1 + \mu^2)(1 - T_2)(\mu^2 + T_2) \\ &= \mu^6(1 - T_2)^2. \end{aligned}$$

Thus,

$$\begin{aligned} S_4 S_4^* &= \frac{\mu^6}{\mu^4(1 + \mu^2)^2} (1 - T_2)^2 \\ &= \frac{\mu^2}{(1 + \mu^2)^2} (1 - T_2)^2. \end{aligned}$$

Applying κ , we have $S_4^* S_4 = \frac{\mu^2}{(1 + \mu^2)^2} (1 - T_2)^2$.

Thus, $S_4^* S_4 = S_4 S_4^* = \frac{\mu^2}{(1 + \mu^2)^2} (1 - T_2)^2$. \square

Lemma 5.3.22. $\mu^2(1 + \mu^2)^2 S_3^* S_3 = (\mu^2 + T_2)[\mu^2(1 + \mu^2) - (1 - T_2)]$.

Proof : Using Lemma 5.3.19 in (5.3.4), we have

$$S_3^* S_3 + T_3^* T_3 + T_3 T_3^* + S_4^* S_4 = 1. \quad (5.3.44)$$

The lemma is derived by substituting the expressions of $T_3^* T_3$, $T_3 T_3^*$ and $S_4^* S_4$ from (5.3.41), (5.3.42) and Lemma 5.3.21 in the equation (5.3.44). \square

Lemma 5.3.23. $(1 + \mu^2)^2 S_3 S_3^* = (1 + \mu^2 T_2)(1 + \mu^2 - \mu^4(1 - T_2))$.

Proof : By unitarity of the matrix Z , in particular equating the (1, 1) th entry of ZZ^* to 1 we get $S_3 S_3^* + \mu^2(1 + \mu^2) T_3 T_3^* + \mu^2 S_4^* S_4 = 1$. Then the Lemma follows by using (5.3.42) and Lemma 5.3.21 in the above equation. \square

Lemma 5.3.24. $-S_2^*S_3 = (\mu^2 + T_2)T_3$.

Proof : By applying $*$ and then multiplying by μ^2 on (5.3.7) we have $\mu^2 S_2^* S_3 + \mu^2 S_4^* S_2 = -\mu^2 T_3(\mu^2 + T_2) + \mu^2(1 - T_2)T_3$. Subtracting this from (5.3.8) we have $(1 - \mu^2)S_2^* S_3 = -T_2 T_3 - \mu^2 T_3 T_2 + \mu^2 T_3(\mu^2 + T_2) - \mu^2(1 - T_2)T_3$ which implies $-S_2^* S_3 = (\mu^2 + T_2)T_3$ as $\mu^2 \neq 1$. \square

Lemma 5.3.25. $S_2(1 - T_2) = \mu^2(1 - T_2)S_2$.

Proof : Applying κ to Lemma 5.3.24 and then taking adjoint, we have

$$\mu^2(1 + \mu^2)^2 T_3^* S_3 = -(\mu^2 + T_2)S_2. \quad (5.3.45)$$

Adding (5.3.37) and (5.3.38) and then taking adjoint, we get (by using $\mu^2 \neq 1$)

$$\mu^2(1 + \mu^2)^2 T_3 S_4 = \mu^4(1 - T_2)S_2. \quad (5.3.46)$$

Moreover, (5.3.27) gives

$$\mu^2(1 + \mu^2)^2 S_4 T_3 = -S_2(\mu^2 + T_2) + (1 - T_2)S_2 - \mu^2(1 + \mu^2)^2 T_3^* S_3.$$

Using (5.3.45), the right hand side of this equation turns out to be $S_2(1 - T_2)$.

Thus,

$$(1 + \mu^2)^2 S_4 T_3 = \mu^{-2} S_2(1 - T_2). \quad (5.3.47)$$

Again, application of adjoint to the equation (5.3.37) gives :

$$\mu^2(1 + \mu^2)^2 S_3 T_3^* = -\mu^2(1 + \mu^2)^2 T_3 S_4 - \mu^2(1 + \mu^2)S_2 + \mu^4(1 - T_2)S_2 + \mu^4 S_2(1 - T_2).$$

Using (5.3.46), we get

$$(1 + \mu^2)^2 S_3 T_3^* = -S_2(1 + \mu^2 T_2). \quad (5.3.48)$$

Using (5.3.45) - (5.3.48) to the equation (5.3.11), we obtain :

$$S_2 - 2S_2 T_2 - (1 + \mu^2)^{-1} \mu^2 S_2(1 + \mu^2 T_2) + \mu^{-2}(1 + \mu^2)^{-1} S_2(1 - T_2) = \mu^2 S_2 - 2\mu^2 T_2 S_2 + (1 + \mu^2)^{-1} \mu^6(1 - T_2)S_2 - (1 + \mu^2)^{-1}(\mu^2 + T_2)S_2.$$

This gives

$$\mu^2(1 + \mu^2)[(S_2 - S_2 T_2) - (\mu^2 S_2 - \mu^2 T_2 S_2)] - \mu^2(1 + \mu^2)(S_2 T_2 - \mu^2 T_2 S_2) - \mu^4 S_2 - \mu^6 S_2 T_2 + S_2(1 - T_2) - \mu^8(S_2 - T_2 S_2) + \mu^4 S_2 + \mu^2 T_2 S_2 = 0.$$

$$\text{Thus, } \mu^2(1 + \mu^2)[S_2(1 - T_2) - \mu^2(1 - T_2)S_2] + S_2(1 - T_2) - \mu^2(S_2 - T_2 S_2) + \mu^6[S_2(1 - T_2) - \mu^2(1 - T_2)S_2] - \mu^6(1 - T_2)S_2 + \mu^4 S_2(1 - T_2) + \mu^2(S_2(1 - T_2) - \mu^2(1 - T_2)S_2) = 0.$$

On simplifying, $(\mu^6 + 2\mu^4 + 2\mu^2 + 1)(S_2(1 - T_2) - \mu^2(1 - T_2)S_2) = 0$, which proves the lemma as $0 < \mu < 1$. \square

Lemma 5.3.26.

$$T_3(1 - T_2) = \mu^2(1 - T_2)T_3, \quad (5.3.49)$$

$$S_3S_4^* = \mu^4S_4^*S_3. \quad (5.3.50)$$

Proof : The equation (5.3.49) follows by applying κ on Lemma 5.3.25 and then taking $*$.

We have $S_4^*S_3 = -T_3^2$ from (5.3.9). On the other hand we have $S_3S_4^* = -\mu^4T_3^2$ from (5.3.22). Combining these two, we get (5.3.50). \square

Lemma 5.3.27. $S_4T_2 = T_2S_4$.

Proof : Subtracting (5.3.33) from (5.3.13) we get the required result. \square

Lemma 5.3.28. $T_3S_2 = S_2T_3$.

Proof : By applying $*$ on (5.3.32) and then subtracting it from (5.3.12) we obtain $S_2T_3 - T_3S_2 = 0$. \square

Lemma 5.3.29. $S_3(1 - T_2) = \mu^4(1 - T_2)S_3$.

Proof : By adding (5.3.12) with (5.3.14) we obtain

$$S_3(1 - T_2) + \mu^2S_3(T_2 - 1) = \mu^2(\mu^2 - 1)T_3S_2.$$

Thus, using $\mu^2 \neq 1$,

$$S_3(1 - T_2) = -\mu^2T_3S_2. \quad (5.3.51)$$

Moreover, by applying $*$ on (5.3.32), we obtain

$$\mu^2(1 - T_2)S_3 = \mu^2S_2T_3 - T_3S_2 + S_3(1 - T_2).$$

Thus,

$$\mu^4(1 - T_2)S_3 = \mu^4S_2T_3 - \mu^2T_3S_2 + \mu^2S_3(1 - T_2).$$

Hence, to prove the Lemma it suffices to prove:

$$S_3(1 - T_2) = \mu^4S_2T_3 - \mu^2T_3S_2 + \mu^2S_3(1 - T_2).$$

After using $T_3S_2 = S_2T_3$ obtained from Lemma 5.3.28 we get this to be the same as $(1 - \mu^2)S_3(1 - T_2) = \mu^2(\mu^2 - 1)T_3S_2$. This is equivalent to $S_3(1 - T_2) = -\mu^2T_3S_2$ (as $\mu^2 \neq 1$) which follows from (5.3.51). \square

Proposition 5.3.30. *Assume $\mu \neq 1$. The map $SO_\mu(3) \rightarrow \mathcal{Q}$ sending M, L, G, N, C to $-(1+\mu^2)^{-1}S_2, S_3, -\mu^{-1}S_4, (1+\mu^2)^{-1}(1-T_2), \mu T_3$ respectively is a $*$ homomorphism.*

Proof : Now, we note that the proof of this Lemma reduces to verification of the relations on \mathcal{Q} as derived in Lemmas 5.3.19 - 5.3.29 along with the following equations :

$$S_3S_4 = \mu^4S_4S_3, \quad (5.3.52)$$

$$S_3S_2 = \mu^2S_2S_3, \quad (5.3.53)$$

$$S_2S_4 = \mu^2S_4S_2, \quad (5.3.54)$$

$$S_3S_4 = -\frac{\mu^2}{(1+\mu^2)^2}S_2^2, \quad (5.3.55)$$

which follow from Remark 5.3.18, (5.3.30), (5.3.31), (5.3.36) respectively. \square

Theorem 5.3.31. *For $\mu \neq 1$, $QISO_R^+(\mathcal{O}(S_{\mu,c}^2), \mathcal{H}, D) \cong SO_\mu(3)$.*

Proof : We have seen in Theorem 5.2.3 that $SU_\mu(2)$ is an object in $\mathbf{Q}'_{\mathbf{R}}(D)$ and $SO_\mu(3)$ is the corresponding maximal Woronowicz subalgebra for which the action is faithful. Thus, $SO_\mu(3)$ is a quantum subgroup of $QISO_R^+(D)$. Now, Proposition 5.3.30 implies that $QISO_R^+(D)$ is a quantum subgroup of $SO_\mu(3)$, thereby completing the proof. \square

Remark 5.3.32. *We observe that in the proof of Theorem 5.3.31, the only place where the structure of D was used was in Proposition 5.3.8 and there we used the fact that the unitary commutes with $|D|$. Thus, if we replace this spectral triple by $(\mathcal{O}(S_{\mu,c}^2), \mathcal{H}, |D|)$, everything remains same and we deduce that*

$$QISO_R^+(\mathcal{O}(S_{\mu,c}^2), \mathcal{H}, |D|) \cong QISO_R^+(\mathcal{O}(S_{\mu,c}^2), \mathcal{H}, D) \cong SO_\mu(3).$$

5.3.5 The quantum isometry group in the case $\mu = 1$

As we had mentioned earlier, some of the Lemmas which were required for the proof of Theorem 5.3.31 needed the condition $\mu \neq 1$. In this subsection, we will give the proof for $\mu = 1$ case.

To begin with, we prove some of the Lemmas in the case $\mu = 1$ which needed $\mu \neq 1$ previously. We note that the proofs of Lemmas 5.3.19, 5.3.20, 5.3.25, 5.3.26, 5.3.27, 5.3.28 go through even in the case $\mu = 1$. Therefore, we can use these Lemmas here.

Lemma 5.3.33. $S_3(1 - T_2) = (1 - T_2)S_3$.

Proof : From (5.3.34) , we obtain $T_3S_2 + S_3T_2 = T_2S_3 + S_2T_3$.

Using $T_3S_2 = S_2T_3$, from Lemma 5.3.28, we have $S_3T_2 = T_2S_3$ which proves the Lemma. \square

Lemma 5.3.34. $S_2(1 - T_2) = (1 - T_2)S_2$.

Proof : From (5.3.15) and (5.3.16) , we have respectively $S_3T_3 = T_3S_3$ and $S_4T_3^* = T_3^*S_4$. From (5.3.41) and (5.3.42) , we see that in the case $\mu = 1$, T_3 is normal. Hence, $S_3T_3^* = T_3^*S_3$ and $S_4T_3 = T_3S_4$. Using these in (5.3.10) , we have $S_2(1 - T_2) = (1 - T_2)S_2$. \square

Lemma 5.3.35. $S_3S_3^* = S_3^*S_3$.

Proof : Multiplying by 4 the equation (5.3.20) , we have

$$4S_2S_2^* - 4S_3S_3^* - 4S_4S_4^* = -4T_2^2 + 4T_3T_3^* + 4T_3^*T_3.$$

Again, from (5.3.25) , we have

$$16T_3^*T_3 - 4S_3^*S_3 - 4S_4S_4^* = -4T_2^2 + S_2S_2^* + S_2^*S_2.$$

Subtracting this from the previous equation, we have

$$4S_2S_2^* - 4(S_3S_3^* - S_3^*S_3) - 16T_3^*T_3 = 4T_3T_3^* + 4T_3^*T_3 - S_2S_2^* - S_2^*S_2.$$

Again using Lemma 5.3.19, (5.3.41) , (5.3.42) and (5.3.43) in this equation, we obtain

$$-4(S_3S_3^* - S_3^*S_3) = 4(1 - T_2^2) - 6(1 - T_2^2) + 2(1 - T_2^2) = 0,$$

which implies $S_3^*S_3 = S_3S_3^*$. \square

Lemma 5.3.36. $S_4^*S_4 = S_4S_4^*$.

Proof : From (5.3.5) , we have

$$-2S_2^*S_2 + 2S_3^*S_3 + 2S_4^*S_4 = 2T_2^2 - 2T_3T_3^* - 2T_3^*T_3.$$

Again, from (5.3.18) , we obtain

$$-2S_2S_2^* + 2S_3S_3^* + 2S_4S_4^* = 2T_2 - 2(1 - T_2)T_2 - 2T_3T_3^* - 2T_3^*T_3.$$

Subtracting this from the previous equation and using Lemma 5.3.19, (5.3.43) and Lemma 5.3.35 we get $2(S_4^*S_4 - S_4S_4^*) = 0$ implying $S_4^*S_4 = S_4S_4^*$. \square

Now we prove that $QISO_R^+(\mathcal{O}(S_{1,c}^2), \mathcal{H}, D)$ is commutative as a C^* algebra. As $QISO_R^+(\mathcal{O}(S_{1,c}^2), \mathcal{H}, D)$ is generated by T_2, T_3, S_2, S_3, S_4 , it is enough to show that these elements belong to the centre of $QISO_R^+(\mathcal{O}(S_{\mu,c}^2), \mathcal{H}, D)$.

Lemma 5.3.37. T_2, T_3, S_2, S_3, S_4 , belong to the centre of $QISO_R^+(\mathcal{O}(S_{\mu,c}^2), \mathcal{H}, D)$.

Proof : T_2 is self adjoint. From (5.3.41) and (5.3.42) we note that T_3 is normal in the case $\mu = 1$. From Lemma 5.3.19 and (5.3.43), we deduce that S_2 is normal in the case $\mu = 1$. Similarly, from Lemma 5.3.35 and Lemma 5.3.36 we obtain that S_3 and S_4 are normal. Hence, it is enough to show that the elements T_2, T_3, S_2, S_3, S_4 commute among themselves.

Now, from (5.3.49), Lemma 5.3.34, Lemma 5.3.33, Lemma 5.3.27 we get that T_2 commutes with T_3, S_2, S_3, S_4 respectively. From Lemma 5.3.28 and (5.3.15) we have that T_3 commutes with S_2, S_3 respectively. Now taking adjoint on the equation (5.3.16) we obtain that T_3 commutes with S_4^* implying that T_3 commutes with S_4 . From (5.3.30) and (5.3.31), we have that S_2 commutes with S_3 and S_4 respectively. Finally, $S_3 S_4 = S_4 S_3$ follows from Remark 5.3.18. \square

From Lemma 5.3.37, we deduce the following.

Theorem 5.3.38. $QISO_R^+(\mathcal{O}(S_{1,c}^2), \mathcal{H}, D)$ is commutative as a C^* algebra and hence coincides with the classical compact quantum group of orientation preserving isometries of the sphere, that is, $C(SO(3))$.

Remark 5.3.39. Our characterization of $SO_\mu(3)$ as the quantum isometry group of a noncommutative Riemannian manifold generalizes the classical description of the group $SO(3)$ as the group of orientation preserving isometries of the usual Riemannian structure on the 2-sphere. It may be mentioned here that in a very recent article ([54]), P. M. Soltan has characterized $SO_\mu(3)$ as the universal compact quantum group acting on the finite dimensional C^* -algebra $M_2(\mathbb{C})$ such that the action preserves a functional ω_μ defined in [54]. In the classical case, we have three equivalent descriptions of $SO(3)$: (a) as a quotient of $SU(2)$, (b) as the group of (orientation preserving) isometries of S^2 , and (c) as the automorphism group of M_2 . In the quantum case the definition of $SO_\mu(3)$ is an analogue of (a), so the characterization of $SO_\mu(3)$ obtained in this paper as the quantum isometry group, together with Soltan's characterization, completes the generalization of all three classical descriptions of $SO(3)$.

5.3.6 Existence of $\widetilde{QISO}^+(D)$

For the above spectral triple we have been unable to settle the issue of the existence of $\widetilde{QISO}^+(D)$ which is the universal object (if it exists) in the category $\mathbf{Q}'(\mathbf{D})$ mentioned in subsection 3.2.2. Nevertheless, we now show that if it exists, the Woronowicz

subalgebra $QISO^+(D)$ must be $SO_\mu(3)$. In particular, the universal object in the subcategory of CQG s acting by orientation preserving isometries and containing $SO_\mu(3)$ as a quantum subgroup exists.

Lemma 5.3.40. *If $\widetilde{QISO^+(D)}$ exists, its induced action on $S_{\mu,c}^2$, say α_0 , must preserve the state h on the subspace spanned by $\{1, A, B, B^*, AB, AB^*, A^2, B^2, B^{*2}\}$.*

Proof : Let $\mathcal{W}_0 = \mathbb{C}.1$, $\mathcal{W}_{\frac{1}{2}} = \text{Span}\{1, A, B, B^*\}$,
 $\mathcal{W}_{\frac{3}{2}} = \text{Span}\{1, A, B, B^*, AB, AB^*, A^2, B^2, B^{*2}\}$.

We note that the proof of Proposition 5.3.8 and the Lemmas preceding it do not use the assumption that the action is volume preserving, so the proof of Proposition 5.3.8 goes through verbatim implying that α_0 keeps invariant the subspace spanned by $\{1, A, B, B^*\}$ and hence it preserves $\mathcal{W}_{\frac{3}{2}}$ as well. Let $\mathcal{W}_{\frac{3}{2}} = \mathcal{W}_{\frac{1}{2}} \oplus \mathcal{W}'$ be the orthogonal decomposition w.r.t. the Haar state (say h_0) of $QISO^+(D)$. Since $SO_\mu(3)$ is a sub-object of $QISO^+(D)$, there is a CQG morphism π from $QISO^+(D)$ onto $SO_\mu(3)$ satisfying $(\text{id} \otimes \pi)\alpha_0 = \Delta$, where Δ is the $SO_\mu(3)$ action on $S_{\mu,c}^2$. It follows from this that any $QISO^+(D)$ -invariant subspace (in particular \mathcal{W}') is also $SO_\mu(3)$ -invariant. On the other hand, it is easily seen that on $\mathcal{W}_{\frac{3}{2}}$, the $SO_\mu(3)$ -action decomposes as $\mathcal{W}_{\frac{1}{2}} \oplus \mathcal{W}''$, (orthogonality w.r.t. h , the Haar state of $SO_\mu(3)$), where \mathcal{W}'' is a five dimensional irreducible subspace.

We claim that $\mathcal{W}' = \mathcal{W}''$, which will prove that the $QISO^+(D)$ -action α_0 has the same h -orthogonal decomposition as the $SO_\mu(3)$ -action on $\mathcal{W}_{\frac{3}{2}}$, so preserves $\mathbb{C}.1$ and its h -orthogonal complements. This will prove that α_0 preserves the Haar state h on $\mathcal{W}_{\frac{3}{2}}$.

We now prove the claim. We observe that $\mathcal{V} := \mathcal{W}' \cap \mathcal{W}''$ is invariant under the $SO_\mu(3)$ -action but due to the irreducibility of Δ on the vector space \mathcal{W}' or \mathcal{W}'' , it has to be zero or $\mathcal{W}' = \mathcal{W}''$. Now, $\dim(\mathcal{V}) = 0$ implies $\dim(\mathcal{W}') + \dim(\mathcal{W}'') = 5 + 5 > 9 = \dim(\mathcal{W}_{\frac{3}{2}})$ which is a contradiction unless $\mathcal{W}' = \mathcal{W}''$. □

Theorem 5.3.41. *If $\widetilde{QISO^+(D)}$ exists, then we have $QISO^+(D) \cong SO_\mu(3)$. In particular, the universal object in the subcategory of $\mathbf{Q}'(\mathbf{D})$ with objects containing $SO_\mu(3)$ as a sub-object, exists.*

Proof : In Lemma 5.3.40, it was noted that Proposition 5.3.8 follows as before. We observe that the other Lemmas used to prove Theorem 5.3.31 require the conclusion of Lemma 5.3.40 as the only extra ingredient. □

5.4 The spectral triple by Chakraborty and Pal on $S_{\mu,c}^2, c > 0$

Now, we shall consider another class of spectral triples on the Podles spheres and show that they give rise to completely different quantum groups of (orientation preserving) isometries. Indeed, for these spectral triples, we have been able to prove the existence of \widetilde{QISO}^+ and identify it with the CQG $C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$ where \mathbf{Z}^∞ denotes countably infinite copies of the group of integers.

In this section, we will work with $c > 0$.

5.4.1 The spectral triple

Let us describe the spectral triple on $S_{\mu,c}^2$ introduced and studied in [14].

Let $\mathcal{H}_+ = \mathcal{H}_- = l^2(\mathbb{N} \cup \{0\})$, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Let $\{e_n, n \geq 0\}$ be the canonical orthonormal basis of $\mathcal{H}_+ = \mathcal{H}_-$ and N be the operator defined on it by $N(e_n) = ne_n$.

We recall the irreducible representations π_+ and $\pi_- : \mathcal{H}_\pm \rightarrow \mathcal{H}_\pm$ as in [14].

$$\pi_\pm(A)e_n = \lambda_\pm \mu^{2n} e_n, \quad (5.4.1)$$

$$\pi_\pm(B)e_n = c_\pm(n)^{\frac{1}{2}} e_{n-1}, \quad (5.4.2)$$

where

$$e_{-1} = 0, \quad \lambda_\pm = \frac{1}{2} \pm \left(c + \frac{1}{4}\right)^{\frac{1}{2}}, \quad c_\pm(n) = \lambda_\pm \mu^{2n} - (\lambda_\pm \mu^{2n})^2 + c. \quad (5.4.3)$$

Let $\pi = \pi_+ \oplus \pi_-$ and $D = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$.

Then $(S_{\mu,c}^2, \pi, \mathcal{H}, D)$ is a spectral triple.

We note that the eigenvalues of D are $\{n : n \in \mathbf{Z}\}$ and eigenspace is spanned by $\begin{pmatrix} e_n \\ e_n \end{pmatrix}$ corresponding to the positive eigenvalue n and $\begin{pmatrix} e_n \\ -e_n \end{pmatrix}$ for the negative eigenvalue $-n$.

Lemma 5.4.1.

$$\pi_+(B^*)(e_n) = c_+(n+1)^{\frac{1}{2}} e_{n+1},$$

$$\pi_-(B^*)(e_n) = c_-(n+1)^{\frac{1}{2}} e_{n+1}.$$

$$\begin{aligned} \text{Proof : } \left\langle \pi(B) \left(\sum_n c_n \begin{pmatrix} e_n \\ 0 \end{pmatrix} \right), \begin{pmatrix} e_{n'} \\ 0 \end{pmatrix} \right\rangle &= \sum_n c_n c_+(n)^{\frac{1}{2}} \langle e_{n-1}, e_{n'} \rangle = c_{n'+1} \\ c_+(n'+1)^{\frac{1}{2}} &= \sum_n c_n c_+(n'+1)^{\frac{1}{2}} \langle e_n, e_{n'+1} \rangle = \sum_n c_n \end{aligned}$$

$$\left\langle e_n, \overline{c_+(n'+1)^{\frac{1}{2}} e_{n'+1}} \right\rangle = \left\langle \sum_n c_n \begin{pmatrix} e_n \\ 0 \end{pmatrix}, \begin{pmatrix} c_+(n'+1)^{\frac{1}{2}} e_{n'+1} \\ 0 \end{pmatrix} \right\rangle.$$

Hence, $\pi_+(B^*)(e_n) = c_+(n+1)^{\frac{1}{2}} e_{n+1}$.

Similarly, $\pi_-(B^*)(e_n) = c_-(n+1)^{\frac{1}{2}} e_{n+1}$. \square

Lemma 5.4.2. *If P_n, Q_n denote the projections onto the subspace generated by $\begin{pmatrix} e_n \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e_n \end{pmatrix}$ respectively, then P_n, Q_n belong to $\pi(S_{\mu,c}^2)$.*

Proof : We claim that for all $n \neq 0$, $c_+(n)$ and $c_-(n)$ are distinct.

Let $c_+(n) = c_-(n)$. Therefore, $\lambda_+ \mu^{2n} - (\lambda_+ \mu^{2n})^2 + c = \lambda_- \mu^{2n} - (\lambda_- \mu^{2n})^2 + c$. This implies $(\lambda_+ + \lambda_-) \mu^{2n} = 1$. Thus, $\mu^{2n} = 1$ and so n has to be 0.

Now, for all $n \geq 1$, $\pi(B^*B) \begin{pmatrix} e_n \\ 0 \end{pmatrix} = c_+(n)^{\frac{1}{2}} \pi(B^*) \begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} = c_+(n) \begin{pmatrix} e_n \\ 0 \end{pmatrix}$.

Similarly, for all $n \geq 1$, $\pi(B^*B) \begin{pmatrix} 0 \\ e_n \end{pmatrix} = c_-(n) \begin{pmatrix} 0 \\ e_n \end{pmatrix}$.

Hence, for all $n \geq 1$, $c_+(n)$ and $c_-(n)$ is a discrete distinct set of eigenvalues of B^*B with eigenspace spanned by $\begin{pmatrix} e_n \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e_n \end{pmatrix}$ respectively. Hence, the eigenprojections corresponding to these eigenvalues belong to the $C^*(B^*B) \subseteq \pi(S_{\mu,c}^2)$. Hence, P_n, Q_n belong to $\pi(S_{\mu,c}^2)$ for all $n \geq 1$.

Moreover, $\pi(A) \begin{pmatrix} e_0 \\ 0 \end{pmatrix} = \lambda_+ \begin{pmatrix} e_0 \\ 0 \end{pmatrix}$ and $\pi(A) \begin{pmatrix} 0 \\ e_0 \end{pmatrix} = \lambda_- \begin{pmatrix} 0 \\ e_0 \end{pmatrix}$.

Thus, by the same reasons as above, P_0, Q_0 belong to $\pi(S_{\mu,c}^2)$. \square

Lemma 5.4.3. $\pi(S_{\mu,c}^2)'' = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \text{ belong to } \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) : X_{12} = X_{21} = 0 \right\}$.

Proof : It suffices to prove that the commutant $\pi(S_{\mu,c}^2)'$ is the Von Neumann algebra of operators of the form $\left\{ \begin{pmatrix} c_1 I & 0 \\ 0 & c_2 I \end{pmatrix} \right\}$ for some c_1, c_2 in \mathbb{C} . We use the fact that π_+ and π_- are irreducible representations.

Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \pi(S_{\mu,c}^2)'$. Using the fact that X commutes with $\pi(A)$, $\pi(B)$, $\pi(B^*)$, we have: X_{11} belongs to $\pi_+(S_{\mu,c}^2)' \cong \mathbb{C}$ and X_{22} belongs to $\pi_-(S_{\mu,c}^2)' \cong \mathbb{C}$, so let $X_{11} = c_1 I$, $X_{22} = c_2 I$ for some c_1, c_2 .

Moreover,

$$X_{12} \pi_-(A) = \pi_+(A) X_{12}, \quad (5.4.4)$$

$$X_{12} \pi_-(B) = \pi_+(B) X_{12}, \quad (5.4.5)$$

$$X_{12}\pi_-(B^*) = \pi_+(B^*)X_{12}. \quad (5.4.6)$$

Now, (5.4.5) implies $X_{12}e_0$ belongs to $\text{Ker}(\pi_+(B)) = \mathbb{C}e_0$. Let $X_{12}e_0 = p_0e_0$.

We have, $\pi_+(B)(X_{12}e_1) = c_-^{\frac{1}{2}}(1)X_{12}e_0 = p_0c_-^{\frac{1}{2}}(1)e_0$, that is, $\pi_+(B)(X_{12}e_1)$ belongs to $\mathbb{C}e_0$.

Since it follows from the definition of $\pi_+(B)$ that $\pi_+(B)$ maps $\overline{\text{span}\{e_i : i \geq 2\}}$ to $(\mathbb{C}e_0)^\perp = \overline{\text{span}\{e_i : i \geq 1\}}$, $X_{12}e_1$ must belong to $\text{span}\{e_0, e_1\}$.

Inductively, we conclude that for all n , $X_{12}(e_n)$ belongs to $\text{span}\{e_0, e_1, \dots, e_n\}$.

Using the definition of $\pi_\pm(B^*)e_n$ along with (5.4.6), we have $c_-^{\frac{1}{2}}(1)X_{12}e_1 = p_0c_+^{\frac{1}{2}}(1)e_1$, that is, $X_{12}e_1 = p_0\frac{c_+^{\frac{1}{2}}(1)}{c_-^{\frac{1}{2}}(1)}e_1$.

We argue in a similar way by induction that $X_{12}e_n = c'_ne_n$ for some constants c'_n .

Now we apply (5.4.6) and (5.4.5) on the vectors e_n and e_{n+1} to get

$$c'_{n+1} = \frac{c'_nc_+^{\frac{1}{2}}(n+1)}{c_-^{\frac{1}{2}}(n+1)} \text{ and } c'_{n+1} = \frac{c'_nc_+^{\frac{1}{2}}(n+1)}{c_+^{\frac{1}{2}}(n+1)}.$$

Since $c_+(n+1) \neq c_-(n+1)$ for $n \geq 0$, we have $c'_n = 0$.

Hence, $c'_n = 0$ for all n implying $X_{12} = 0$.

It follows similarly that $X_{21} = 0$. □

5.4.2 Computation of the quantum isometry group

Let $(\tilde{\mathcal{Q}}, \Delta, U)$ be an object in the category $\mathbf{Q}'(D)$, with $\alpha = \alpha_U$ and the corresponding Woronowicz C^* subalgebra of $\tilde{\mathcal{Q}}$ generated by $\{<(\xi \otimes 1), \alpha(x)(\eta \otimes 1)>_{\tilde{\mathcal{Q}}}, \xi, \eta \in \mathcal{H}, x \in S_{\mu,c}^2\}$ is denoted by \mathcal{Q} . Assume, without loss of generality, that the representation U is faithful.

Since U commutes with D , it preserves the eigenvectors $\begin{pmatrix} e_n \\ e_n \end{pmatrix}$ and $\begin{pmatrix} e_n \\ -e_n \end{pmatrix}$.

$$\text{Let } U \begin{pmatrix} e_n \\ e_n \end{pmatrix} = \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes q_n^+,$$

$$U \begin{pmatrix} e_n \\ -e_n \end{pmatrix} = \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes q_n^-,$$

for some q_n^+, q_n^- in $\tilde{\mathcal{Q}}$.

Lemma 5.4.4. For all $n \geq 0$, $\alpha(A) \begin{pmatrix} e_n \\ 0 \end{pmatrix} = \begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \frac{1}{4}\{\lambda_+\mu^{2n}(1 + q_n^+q_n^{-*}) + \lambda_-\mu^{2n}(1 - q_n^+q_n^{-*}) + \lambda_+\mu^{2n}(1 + q_n^-q_n^{+*}) - \lambda_-\mu^{2n}(q_n^-q_n^{+*} - 1)\} + \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes \frac{1}{4}\{\lambda_+\mu^{2n}(1 + q_n^+q_n^{-*}) + \lambda_-\mu^{2n}(1 - q_n^+q_n^{-*}) - \lambda_+\mu^{2n}(1 + q_n^-q_n^{+*}) + \lambda_-\mu^{2n}(q_n^-q_n^{+*} - 1)\}.$

For all $n \geq 0$, $\alpha(A) \begin{pmatrix} 0 \\ e_n \end{pmatrix} = \begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \frac{1}{4}[\lambda_+ \mu^{2n}(1 - q_n^+ q_n^{-*}) + \lambda_- \mu^{2n}(1 + q_n^+ q_n^{-*}) + \lambda_+ \mu^{2n}(q_n^- q_n^{+*} - 1) - \lambda_- \mu^{2n}(q_n^- q_n^{+*} + 1)] + \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes \frac{1}{4}[\lambda_+ \mu^{2n}(1 - q_n^+ q_n^{-*}) + \lambda_- \mu^{2n}(1 + q_n^+ q_n^{-*}) - \lambda_+ \mu^{2n}(q_n^- q_n^{+*} - 1) + \lambda_- \mu^{2n}(q_n^- q_n^{+*} + 1)]$.

For all $n \geq 1$, $\alpha(B) \begin{pmatrix} e_n \\ 0 \end{pmatrix} = \begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} \otimes \frac{1}{4}[(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})q_{n-1}^+ q_n^{+*} + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})q_{n-1}^- q_n^{+*} + (c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})q_{n-1}^+ q_n^{-*} + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})q_{n-1}^- q_n^{-*}] + \begin{pmatrix} 0 \\ e_{n-1} \end{pmatrix} \otimes \frac{1}{4}[(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})q_{n-1}^+ q_n^{+*} - (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})q_{n-1}^- q_n^{+*} + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})q_{n-1}^+ q_n^{-*} - (c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})q_{n-1}^- q_n^{-*}]$.

For all $n \geq 1$, $\alpha(B) \begin{pmatrix} 0 \\ e_n \end{pmatrix} = \begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} \otimes \frac{1}{4}[(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})q_{n-1}^+ q_n^{+*} + (c_-(n)^{\frac{1}{2}} - c_+(n)^{\frac{1}{2}})q_{n-1}^- q_n^{+*} - (c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})q_{n-1}^+ q_n^{-*} + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})q_{n-1}^- q_n^{-*}] + \begin{pmatrix} 0 \\ e_{n-1} \end{pmatrix} \otimes \frac{1}{4}[(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})q_{n-1}^+ q_n^{+*} + (c_-(n)^{\frac{1}{2}} - c_+(n)^{\frac{1}{2}})q_{n-1}^- q_n^{+*} - (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})q_{n-1}^+ q_n^{-*} + (c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})q_{n-1}^- q_n^{-*}]$.

For all $n \geq 0$, $\alpha(B^*) \begin{pmatrix} e_n \\ 0 \end{pmatrix} = \begin{pmatrix} e_{n+1} \\ 0 \end{pmatrix} \otimes \frac{1}{4}[(c_+(n+1)^{\frac{1}{2}} + c_-(n+1)^{\frac{1}{2}})(q_{n+1}^+ q_n^{+*} + q_{n+1}^- q_n^{-*}) + (c_+(n+1)^{\frac{1}{2}} - c_-(n+1)^{\frac{1}{2}})(q_{n+1}^+ q_n^{-*} + q_{n+1}^- q_n^{+*})] + \begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix} \otimes \frac{1}{4}[(c_+(n+1)^{\frac{1}{2}} + c_-(n+1)^{\frac{1}{2}})(q_{n+1}^+ q_n^{+*} - q_{n+1}^- q_n^{-*}) + (c_+(n+1)^{\frac{1}{2}} - c_-(n+1)^{\frac{1}{2}})(q_{n+1}^+ q_n^{-*} - q_{n+1}^- q_n^{+*})]$.

Proof : One has, $\alpha(A) \begin{pmatrix} e_n \\ 0 \end{pmatrix}$
 $= \tilde{U}(A \otimes 1) \tilde{U}^* \left(\begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes 1 \right)$
 $= \frac{1}{2} \tilde{U}(\pi(A) \otimes 1) \tilde{U}^* \left(\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes 1 + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes 1 \right)$
 $= \frac{1}{2} \tilde{U}(\pi(A) \otimes 1) \left[\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes q_n^{+*} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes q_n^{-*} \right]$
 $= \frac{1}{2} \tilde{U} \left[\begin{pmatrix} \pi_+(A) e_n \\ \pi_-(A) e_n \end{pmatrix} \otimes q_n^{+*} + \begin{pmatrix} \pi_+(A) e_n \\ -\pi_-(A) e_n \end{pmatrix} \otimes q_n^{-*} \right]$
By using (5.4.1), we get this to be equal to
 $= \frac{1}{2} \tilde{U} \left[\begin{pmatrix} \lambda_+ \mu^{2n} e_n \\ \lambda_- \mu^{2n} e_n \end{pmatrix} \otimes q_n^{+*} + \begin{pmatrix} \lambda_+ \mu^{2n} e_n \\ -\lambda_- \mu^{2n} e_n \end{pmatrix} \otimes q_n^{-*} \right]$

$$\begin{aligned}
&= \frac{1}{2} \tilde{U} \left[\begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \lambda_+ \mu^{2n} (q_n^{+*} + q_n^{-*}) + \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes \lambda_- \mu^{2n} (q_n^{+*} - q_n^{-*}) \right] \\
&= \frac{1}{4} \tilde{U} \left[\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes \{ \lambda_+ \mu^{2n} (q_n^{+*} + q_n^{-*}) + \lambda_- \mu^{2n} (q_n^{+*} - q_n^{-*}) \} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes \{ \lambda_+ \mu^{2n} (q_n^{+*} + \right. \\
&\quad \left. q_n^{-*}) - \lambda_- \mu^{2n} (q_n^{+*} - q_n^{-*}) \} \right] \\
&= \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes \frac{1}{4} q_n^+ \{ \lambda_+ \mu^{2n} (q_n^{+*} + q_n^{-*}) + \lambda_- \mu^{2n} (q_n^{+*} - q_n^{-*}) \} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes \frac{1}{4} q_n^- \{ \lambda_+ \mu^{2n} (q_n^{+*} + \\
&\quad q_n^{-*}) - \lambda_- \mu^{2n} (q_n^{+*} - q_n^{-*}) \} \\
&= \begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \frac{1}{4} \{ \lambda_+ \mu^{2n} (1 + q_n^+ q_n^{-*}) + \lambda_- \mu^{2n} (1 - q_n^+ q_n^{-*}) + \lambda_+ \mu^{2n} (1 + q_n^- q_n^{+*}) - \\
&\quad \lambda_- \mu^{2n} (q_n^- q_n^{+*} - 1) \} + \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes \frac{1}{4} \{ \lambda_+ \mu^{2n} (1 + q_n^+ q_n^{-*}) + \lambda_- \mu^{2n} (1 - q_n^+ q_n^{-*}) - \lambda_+ \mu^{2n} (1 + \\
&\quad q_n^- q_n^{+*}) + \lambda_- \mu^{2n} (q_n^- q_n^{+*} - 1) \}.
\end{aligned}$$

$$\begin{aligned}
&\text{Similarly, } \alpha(A) \begin{pmatrix} 0 \\ e_n \end{pmatrix} \\
&= \tilde{U}(\pi(A) \otimes 1) \tilde{U}^* \left(\begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes 1 \right) \\
&= \frac{1}{2} \tilde{U}(\pi(A) \otimes 1) \tilde{U}^* \left[\begin{pmatrix} e_n \\ e_n \end{pmatrix} - \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \right] \otimes 1 \\
&= \frac{1}{2} \tilde{U} \left[\begin{pmatrix} \pi_+(A) e_n \\ \pi_-(A) e_n \end{pmatrix} \otimes q_n^{+*} - \begin{pmatrix} \pi_+(A) e_n \\ -\pi_-(A) e_n \end{pmatrix} \otimes q_n^{-*} \right] \\
&= \frac{1}{2} \tilde{U} \left[\begin{pmatrix} \lambda_+ \mu^{2n} e_n \\ \lambda_- \mu^{2n} e_n \end{pmatrix} \otimes q_n^{+*} - \begin{pmatrix} \lambda_+ \mu^{2n} e_n \\ -\lambda_- \mu^{2n} e_n \end{pmatrix} \otimes q_n^{-*} \right] \\
&= \frac{1}{4} \tilde{U} \left[\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes \{ \lambda_+ \mu^{2n} (q_n^{+*} - q_n^{-*}) + \lambda_- \mu^{2n} (q_n^{+*} + q_n^{-*}) \} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes \{ \lambda_+ \mu^{2n} (q_n^{+*} - \right. \\
&\quad \left. q_n^{-*}) - \lambda_- \mu^{2n} (q_n^{+*} + q_n^{-*}) \} \right] \\
&= \begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \frac{1}{4} [\lambda_+ \mu^{2n} (1 - q_n^+ q_n^{-*}) + \lambda_- \mu^{2n} (1 + q_n^+ q_n^{-*}) + \lambda_+ \mu^{2n} (q_n^- q_n^{+*} - 1) - \\
&\quad \lambda_- \mu^{2n} (q_n^- q_n^{+*} + 1)] + \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes \frac{1}{4} [\lambda_+ \mu^{2n} (1 - q_n^+ q_n^{-*}) + \lambda_- \mu^{2n} (1 + q_n^+ q_n^{-*}) - \lambda_+ \mu^{2n} (q_n^- q_n^{+*} - \\
&\quad 1) + \lambda_- \mu^{2n} (q_n^- q_n^{+*} + 1)].
\end{aligned}$$

As the proof of the others are exactly similar, we omit the proofs.

Lemma 5.4.5. *We have:*

$$q_n^+ q_n^{-*} = q_n^- q_n^{+*} \quad \text{for all } n, \quad (5.4.7)$$

$$(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} - q_{n-1}^- q_n^{-*}) + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{-*} - q_{n-1}^- q_n^{+*}) = 0$$

$$\text{for all } n \geq 1, \quad (5.4.8)$$

$$(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} - q_{n-1}^- q_n^{-*}) + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^- q_n^{+*} - q_{n-1}^+ q_n^{-*}) = 0$$

for all $n \geq 1$,

(5.4.9)

$$(c_+(n+1)^{\frac{1}{2}} + c_-(n+1)^{\frac{1}{2}})(q_{n+1}^+ q_n^{+*} - q_{n+1}^- q_n^{-*}) = (c_+(n+1)^{\frac{1}{2}} - c_-(n+1)^{\frac{1}{2}})(q_{n+1}^- q_n^{+*} - q_{n+1}^+ q_n^{-*})$$

for all n .

(5.4.10)

Proof : Since $\alpha(A)$ maps $\pi(S_{\mu,c}^2)$ into its double commutant, we conclude by using the description of $\pi(S_{\mu,c}^2)''$ given in Lemma 5.4.3 that the coefficient of $\begin{pmatrix} 0 \\ e_n \end{pmatrix}$ in

$$\alpha(A) \begin{pmatrix} e_n \\ 0 \end{pmatrix} \text{ must be 0, which implies (by Lemma 5.4.4)}$$

$$\lambda_+[1 + q_n^+ q_n^{-*} - (1 + q_n^- q_n^{+*})] + \lambda_-[1 - q_n^+ q_n^{-*} + q_n^- q_n^{+*} - 1] = 0.$$

Hence,

$$(\lambda_+ - \lambda_-)(q_n^+ q_n^{-*} - q_n^- q_n^{+*}) = 0. \text{ Hence, } (q_n^+ q_n^{-*} - q_n^- q_n^{+*}) = 0.$$

Proceeding in a similar way, (5.4.8), (5.4.9), (5.4.10) follow from the facts that coefficients of $\begin{pmatrix} 0 \\ e_{n-1} \end{pmatrix}$, $\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix}$ in $\alpha(B) \begin{pmatrix} e_n \\ 0 \end{pmatrix}$,

$$\alpha(B) \begin{pmatrix} 0 \\ e_n \end{pmatrix} \text{ and } \alpha(B^*) \begin{pmatrix} e_n \\ 0 \end{pmatrix} \text{ (respectively) are zero.}$$

□

Corollary 5.4.6. *We have*

$$\begin{aligned} \alpha(A) &= \sum_{n=0}^{\infty} AP_n \otimes \frac{1}{2\lambda_+} \{ \lambda_+(1 + q_n^+ q_n^{-*}) + \lambda_-(1 - q_n^+ q_n^{-*}) \} \\ &\quad + \sum_{n=0}^{\infty} AQ_n \otimes \frac{1}{2\lambda_-} \{ \lambda_+(1 - q_n^+ q_n^{-*}) + \lambda_-(1 + q_n^+ q_n^{-*}) \}. \\ \alpha(B) &= \sum_{n=1}^{\infty} BP_n \otimes \frac{1}{4c_+(n)} [(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*}) \\ &\quad + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*})] + \sum_{n=1}^{\infty} BQ_n \otimes \frac{1}{4c_-(n)} [(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}}) \end{aligned}$$

$$(q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*}) - (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{-*} + q_{n-1}^- q_n^{+*})].$$

Proof : We note that $\pi(A) \begin{pmatrix} e_n \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_+(A) & 0 \\ 0 & \pi_-(A) \end{pmatrix} \begin{pmatrix} e_n \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_+(A)e_n \\ 0 \end{pmatrix} = \lambda_+ \mu^{2n} \begin{pmatrix} e_n \\ 0 \end{pmatrix}.$

Thus, $\begin{pmatrix} e_n \\ 0 \end{pmatrix} = \frac{\pi(A) \begin{pmatrix} e_n \\ 0 \end{pmatrix}}{\lambda_+ \mu^{2n}}.$

Similarly, $\begin{pmatrix} 0 \\ e_n \end{pmatrix} = \frac{\pi(A) \begin{pmatrix} 0 \\ e_n \end{pmatrix}}{\lambda_- \mu^{2n}}.$

Now, using (5.4.7),

$$\begin{aligned} \alpha(A) \begin{pmatrix} e_n \\ 0 \end{pmatrix} &= \begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \frac{1}{2} \{ \lambda_+ \mu^{2n} (1 + q_n^+ q_n^{-*}) + \lambda_- \mu^{2n} (1 - q_n^+ q_n^{-*}) \} \\ &= \pi(A) \begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \frac{1}{2\lambda_+} \{ \lambda_+ (1 + q_n^+ q_n^{-*}) + \lambda_- (1 - q_n^+ q_n^{-*}) \}. \end{aligned}$$

Similarly,

$$\alpha(A) \begin{pmatrix} 0 \\ e_n \end{pmatrix} = \pi(A) \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes \frac{1}{2\lambda_-} \{ \lambda_+ (1 - q_n^+ q_n^{-*}) + \lambda_- (1 + q_n^+ q_n^{-*}) \}.$$

Thus, $\alpha(A) = \sum_{n=0}^{\infty} A P_n \otimes \frac{1}{2\lambda_+} \{ \lambda_+ (1 + q_n^+ q_n^{-*}) + \lambda_- (1 - q_n^+ q_n^{-*}) \} + \sum_{n=0}^{\infty} A Q_n \otimes \frac{1}{2\lambda_-} \{ \lambda_+ (1 - q_n^+ q_n^{-*}) + \lambda_- (1 + q_n^+ q_n^{-*}) \}.$

By similar considerations from $\alpha(B) \begin{pmatrix} e_n \\ 0 \end{pmatrix}, \alpha(B) \begin{pmatrix} 0 \\ e_n \end{pmatrix}$, the result follows. \square

Lemma 5.4.7. *Let $\widetilde{P}_n = P_n + Q_n$. Then for each vector v in \mathcal{H} , $\alpha(\widetilde{P}_n)v = \widetilde{P}_n v \otimes 1$.*

Proof : To start with, we recall that P_n and Q_n belong to $\pi(S_{\mu,c}^2)$ (Lemma 5.4.2). Hence, \widetilde{P}_n belongs to $\pi(S_{\mu,c}^2)$.

$$\begin{aligned} \alpha(\widetilde{P}_n) \begin{pmatrix} e_n \\ e_n \end{pmatrix} \\ = \widetilde{U}(\widetilde{P}_n \otimes 1) \widetilde{U}^* \begin{pmatrix} e_n \\ e_n \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \tilde{U}\left(\begin{pmatrix} e_n \\ e_n \end{pmatrix}\right) \otimes q_n^{+*} \\
&= \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes 1 \\
&= (\widetilde{P}_n \otimes 1)\left(\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes 1\right).
\end{aligned}$$

For $k \neq n$,

$$\begin{aligned}
&\tilde{U}(\widetilde{P}_k \otimes 1)\tilde{U}^*\begin{pmatrix} e_n \\ e_n \end{pmatrix} \\
&= \tilde{U}(\widetilde{P}_k \otimes 1)\left(\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes q_n^{+*}\right) \\
&= 0 \\
&= (\widetilde{P}_k \otimes 1)\left(\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes 1\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\alpha(\widetilde{P}_n)\begin{pmatrix} e_n \\ -e_n \end{pmatrix} \\
&= \tilde{U}\left(\begin{pmatrix} e_n \\ -e_n \end{pmatrix}\right) \otimes q_n^{-*} \\
&= \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes 1 \\
&= (\widetilde{P}_n \otimes 1)\left(\begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes 1\right),
\end{aligned}$$

and for

$$k \neq n, \alpha(\widetilde{P}_k)\begin{pmatrix} e_n \\ -e_n \end{pmatrix} = 0 = (\widetilde{P}_k \otimes 1)\left(\begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes 1\right).$$

Combining all these, we get the required result. \square

Proposition 5.4.8. *As a C^* -algebra, $\tilde{\mathcal{Q}}$ is generated by the unitaries $\{q_n^+\}_{n \geq 0}$, and the self-adjoint unitary $y_0 = q_0^{-*}q_0^+$. Moreover, \mathcal{Q} is generated by unitaries $z_n = q_{n-1}^+q_n^{+*}$, $n \geq 1$ along with a self adjoint unitary w' .*

Proof : Replacing $n + 1$ by n in (5.4.10) we have,

$$(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_n^+ q_{n-1}^{+*} - q_n^- q_{n-1}^{-*}) - (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_n^- q_{n-1}^{+*} - q_n^+ q_{n-1}^{-*}) = 0 \text{ for all } n \geq 1. \quad (5.4.11)$$

Subtracting (5.4.11) from the equation obtained by applying $*$ on (5.4.8), we have : $2(c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_n^- q_{n-1}^{+*} - q_n^+ q_{n-1}^{-*}) = 0$ for all $n \geq 1$. Now, from the proof of Lemma 5.4.2, $(c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}}) \neq 0$ for all $n \geq 1$. This implies :

$$q_n^- q_{n-1}^{+*} = q_n^+ q_{n-1}^{-*} \text{ for all } n \geq 1. \quad (5.4.12)$$

Using (5.4.12) in (5.4.11), we have

$$q_n^+ q_{n-1}^{+*} = q_n^- q_{n-1}^{-*} \text{ for all } n \geq 1. \quad (5.4.13)$$

Let $y_n = q_n^{-*} q_n^+$.

Then, using (5.4.7), we have $q_n^{-*} q_n^+ = q_n^{+*} q_n^-$. Hence, $y_n = y_n^*$. Moreover, as y_n is a product of unitaries, it is a self adjoint unitary.

Now, from (5.4.12), we have $q_n^- = q_n^+ y_{n-1}^*$ for all $n \geq 1$. Therefore,

$$q_n^- = q_n^+ y_{n-1} \text{ for all } n \geq 1. \quad (5.4.14)$$

Next, from (5.4.13), we obtain $q_n^{-*} q_n^+ = q_{n-1}^{-*} q_{n-1}^+$ for all $n \geq 1$ implying

$$y_n = y_{n-1} \text{ for all } n \geq 1. \quad (5.4.15)$$

From (5.4.14) and (5.4.15) and the faithfulness of the representation U , we conclude that $\tilde{\mathcal{Q}}$ is generated by $\{q_n^+\}_{n \geq 0}$ and y_0 .

Now we prove the second part of the proposition.

Using Lemma 5.4.7, we note that for all v in \mathcal{H} , $\alpha(A\tilde{P}_k)v = \alpha(A)(\tilde{P}_k v \otimes 1) = AP_k v \otimes \frac{1}{2\lambda_+} \{ \lambda_+(1 + q_k^+ q_k^{-*}) + \lambda_-(1 - q_k^+ q_k^{-*}) \} + AQ_k v \otimes \frac{1}{2\lambda_-} \{ \lambda_+(1 - q_k^+ q_k^{-*}) + \lambda_-(1 + q_k^+ q_k^{-*}) \}$. Therefore, $\alpha(A\tilde{P}_k) = AP_k \otimes \frac{1}{2\lambda_+} \{ \lambda_+(1 + q_k^+ q_k^{-*}) + \lambda_-(1 - q_k^+ q_k^{-*}) \} + AQ_k \otimes \frac{1}{2\lambda_-} \{ \lambda_+(1 - q_k^+ q_k^{-*}) + \lambda_-(1 + q_k^+ q_k^{-*}) \}$.

Now, AP_k and AQ_k being distinct elements, there exist linear functional ϕ such that $\phi(AP_k) = 1$, $\phi(AQ_k) = 0$ and vice versa. Hence, $(\phi \otimes \text{id})\alpha(A\tilde{P}_k) = \lambda_+(1 + q_m^+ q_m^{-*}) + \lambda_-(1 - q_m^+ q_m^{-*})$ belongs to \mathcal{Q} . Similarly, $\lambda_+(1 - q_m^+ q_m^{-*}) + \lambda_-(1 + q_m^+ q_m^{-*})$ belongs to \mathcal{Q} for all m .

Subtracting, we get $q_m^+ q_m^{-*}$ belongs to \mathcal{Q} .

Using the expression of $\alpha(B)$ in a similar way, we have

$(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*}) + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*})$ belongs to \mathcal{Q} for all $n \geq 1$.

and $(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*}) - (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{-*} + q_{n-1}^- q_n^{+*})$ belongs to \mathcal{Q} for all $n \geq 1$.

Adding and subtracting, we have

$$q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*} \in \mathcal{Q} \text{ for all } n \geq 1, \quad (5.4.16)$$

$$q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*} \in \mathcal{Q} \text{ for all } n \geq 1. \quad (5.4.17)$$

Recalling (5.4.15), we have $q_n^- = q_n^+ y_{n-1} = q_n^+ y_0$. Using this in (5.4.16), we obtain

$$\begin{aligned} & q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*} \\ &= q_{n-1}^+ q_n^{+*} + q_{n-1}^+ y_0 y_0^* q_n^{+*} \\ &= q_{n-1}^+ q_n^{+*} + q_{n-1}^+ q_n^{+*} \\ &= 2q_{n-1}^+ q_n^{+*}. \end{aligned}$$

Similarly, using $q_n^- = q_n^+ y_0$ in (5.4.17), one has

$$\begin{aligned} & q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*} \\ &= q_{n-1}^+ y_0 q_n^{+*} + q_{n-1}^+ y_0 q_n^{+*} = 2q_{n-1}^+ y_0 q_n^{+*}. \end{aligned}$$

Hence, we conclude that $q_{n-1}^+ q_n^{+*}$ and $q_{n-1}^+ y_0 q_n^{+*}$ are in \mathcal{Q} for all $n \geq 1$.

Let

$$\begin{aligned} z_n &= q_{n-1}^+ q_n^{+*}, \\ w_n &= q_{n-1}^+ y_0 q_n^{+*}, \end{aligned}$$

for all $n \geq 1$.

Then, we observe that

$$z_n^* w_n = q_n^+ y_0 q_n^{+*} = q_n^+ q_n^{-*}.$$

Moreover,

$$q_0^+ q_0^{-*} = q_0^+ (q_0^+ y_0^*)^* = q_0^+ y_0 q_0^{+*} = q_0^+ y_0 q_1^{+*} q_1^+ q_0^{+*} = w_1 z_1^*.$$

Thus for all $n \geq 0$, $q_n^+ q_n^{-*}$ belong to $C^*(\{z_n, w_n\}_{n \geq 1})$.

For all $n \geq 2$, $q_{n-1}^- q_n^{-*} = q_{n-1}^+ y_{n-2}^* (q_n^+ y_{n-1}^*)^* = q_{n-1}^+ y_0^* y_0 q_n^{+*} = q_{n-1}^+ q_n^{+*} = q_{n-1}^+ q_n^{+*} = z_n$,

$$q_0^- q_1^{-*} = q_0^+ y_0 (q_1^+ y_0)^* = q_0^+ y_0 y_0^* q_1^{+*} = q_0^+ q_1^{+*} = z_1.$$

Finally,

$$q_{n-1}^- q_n^{+*} = q_{n-1}^+ y_0^* q_n^{+*} = w_n$$

and

$$q_0^- q_1^{+*} = q_0^+ y_0 q_1^{+*} = w_1.$$

Now, from the expressions of $\alpha(A)$ and $\alpha(B)$, it is clear that \mathcal{Q} is generated by $q_n^+ q_n^{-*}, q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*}, q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*}$. By the above made observations, these belong to $C^*(\{z_n, w_n\}_{n \geq 1})$ which implies that \mathcal{Q} is a C^* subalgebra of $C^*(\{z_n, w_n\}_{n \geq 1})$. Moreover, from the definitions of z_n, w_n it is clear that $C^*(\{z_n, w_n\}_{n \geq 1})$ is a C^* subalgebra of \mathcal{Q} .

Therefore, $\mathcal{Q} \cong C^*(\{z_n, w_n\}_{n \geq 1})$.

In fact, a simpler description is possible by noting that $z_n w_{n+1} = q_{n-1}^+ q_n^{+*} q_n^+ y_0 q_{n+1}^{+*} = q_{n-1}^+ y_0 q_{n+1}^{+*} = q_{n-1}^+ y_0 q_n^{+*} q_n^+ q_{n+1}^{+*} = w_n z_{n+1}$ and so, $w_{n+1} = z_n^* w_n z_{n+1}$ which implies $\{w_n\}_{n \geq 1}$ is a subset of $C^*(\{z_n\}_{n \geq 1}, w_1)$.

Defining $w' = w_1^* z_1$, we note that $z_1 = q_0^+ y_0 q_1^{+*} q_1^+ y_0^* q_1^{+*} = w_1 (z_1^* w_1)$ which implies $w_1^* z_1 = z_1^* w_1$. Thus, w' is self adjoint. It is a unitary as it is a product of unitaries.

Thus $\mathcal{Q} \cong C^*(\{z_n\}_{n \geq 1}, w')$. \square

Lemma 5.4.9. $\Delta(q_n^\pm) = q_n^\pm \otimes q_n^\pm$,

$$\Delta(y_1) = y_1 \otimes y_1.$$

Proof : We use the fact that U is a unitary representation.

$$(\text{id} \otimes \Delta)U \begin{pmatrix} e_n \\ e_n \end{pmatrix} = (\text{id} \otimes \Delta) \left(\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes q_n^+ \right) = \left(\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes \Delta(q_n^+) \right).$$

$$U_{(12)} U_{(13)} \left(\begin{pmatrix} e_n \\ e_n \end{pmatrix} \right) = \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes q_n^+ \otimes q_n^+.$$

$$\text{Hence, } \Delta(q_n^+) = q_n^+ \otimes q_n^+.$$

$$\text{Similarly, } \Delta(q_n^-) = q_n^- \otimes q_n^-.$$

$$\text{Moreover, } \Delta(y_1) = \Delta(q_n^- q_n^{+*}) = (q_n^- \otimes q_n^-) (q_n^{+*} \otimes q_n^{+*}) = q_n^- q_n^{+*} \otimes q_n^- q_n^{+*} = y_1 \otimes y_1. \quad \square$$

Let us now consider the quantum group $\tilde{\mathcal{S}} \cong C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$, where $\mathbf{Z}^\infty = \mathbf{Z} * \mathbf{Z} * \dots$ denotes the free product of countably infinitely many copies of \mathbf{Z} . By the Remarks 1.1.7, 1.1.4 and 1.1.3, $\tilde{\mathcal{S}} \cong C(\mathbf{Z}_2) * C(\mathbb{T}) * C(\mathbb{T}) * \dots$, and let us denote by r_n^+ the generator of n th copy of $C(\mathbb{T})$ and by y the generator of $C(\mathbf{Z}_2)$.

The coproduct Δ_0 on $\tilde{\mathcal{S}}$ is given by $\Delta_0(r_n^+) = r_n^+ \otimes r_n^+, \Delta_0(y) = y \otimes y$.

Define

$$V \begin{pmatrix} e_n \\ e_n \end{pmatrix} = \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes r_n^+.$$

$$V \begin{pmatrix} e_n \\ -e_n \end{pmatrix} = \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes r_n^+ y.$$

Lemma 5.4.10. *V commutes with D and V is a unitary representation of $\tilde{\mathcal{S}}$, that is $(\tilde{\mathcal{S}}, \Delta_0, V)$ is an object in $\mathbf{Q}'(D)$.*

Proof : As the eigenspaces corresponding to distinct eigenvalues of D are spanned by $\begin{pmatrix} e_n \\ e_n \end{pmatrix}$ and $\begin{pmatrix} e_n \\ -e_n \end{pmatrix}$, V commutes with D .

The fact that V is a representation follows from the proof of Lemma 5.4.9.

To prove that V is a unitary, it is enough to check the following:

$$\begin{aligned} \left\langle V \begin{pmatrix} e_n \\ e_n \end{pmatrix}, V \begin{pmatrix} e_m \\ e_m \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} e_n \\ e_n \end{pmatrix}, \begin{pmatrix} e_m \\ e_m \end{pmatrix} \right\rangle .1, \quad \left\langle V \begin{pmatrix} e_n \\ e_n \end{pmatrix}, V \begin{pmatrix} e_m \\ -e_m \end{pmatrix} \right\rangle = \\ \left\langle \begin{pmatrix} e_n \\ e_n \end{pmatrix}, \begin{pmatrix} e_m \\ -e_m \end{pmatrix} \right\rangle .1, \quad \left\langle V \begin{pmatrix} e_n \\ -e_n \end{pmatrix}, V \begin{pmatrix} e_m \\ -e_m \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} e_n \\ -e_n \end{pmatrix}, \begin{pmatrix} e_m \\ -e_m \end{pmatrix} \right\rangle .1, \end{aligned}$$

The proofs being similar, we prove only the first equation.

$$\begin{aligned} \left\langle V \begin{pmatrix} e_n \\ e_n \end{pmatrix}, V \begin{pmatrix} e_m \\ e_m \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes r_n^+, \begin{pmatrix} e_m \\ e_m \end{pmatrix} \otimes r_m^+ \right\rangle \\ &= \left\langle \begin{pmatrix} e_n \\ e_n \end{pmatrix}, \begin{pmatrix} e_m \\ e_m \end{pmatrix} \right\rangle \otimes r_n^{+*} r_m^+, \end{aligned}$$

$$\text{which, if } n \neq m \text{ equals } 0 = \left\langle \begin{pmatrix} e_n \\ e_n \end{pmatrix}, \begin{pmatrix} e_m \\ e_m \end{pmatrix} \right\rangle .1,$$

$$\text{and if } n = m, \text{ equals } 1 = \left\langle \begin{pmatrix} e_n \\ e_n \end{pmatrix}, \begin{pmatrix} e_m \\ e_m \end{pmatrix} \right\rangle .1. \quad \square$$

Setting $r_n^- = r_n^+ y$, we observe that r_n^- is a unitary and satisfies :

$$r_n^- r_{n-1}^{+*} = r_n^+ r_{n-1}^{-*} \quad \text{for all } n \geq 1. \quad (5.4.18)$$

$$r_n^+ r_{n-1}^{+*} = r_n^- r_{n-1}^{-*} \quad \text{for all } n \geq 1. \quad (5.4.19)$$

Using $r_n^- = r_n^+ r_{n-1}^{-*} r_{n-1}^+$ (from 5.4.18) in (5.4.19) we have $r_{n-1}^+ = r_{n-1}^- r_{n-1}^{+*} r_{n-1}^-$.

This implies

$$r_n^+ r_n^{-*} = r_n^- r_n^{+*} \quad \text{for all } n. \quad (5.4.20)$$

Moreover, taking $*$ on (5.4.18) and (5.4.19) respectively, we get the following:

$$r_{n-1}^+ r_n^{-*} - r_{n-1}^- r_n^{+*} = 0 \quad \text{for all } n \geq 1. \quad (5.4.21)$$

$$r_{n-1}^+ r_n^{+*} - r_{n-1}^- r_n^{-*} = 0 \quad \text{for all } n \geq 1. \quad (5.4.22)$$

Thus, the equations (5.4.7) - (5.4.10) in Lemma 5.4.5 are satisfied with q_n^\pm 's replaced by r_n^\pm 's and hence it is easy to see that there is a C^* -homomorphism from $\tilde{\mathcal{S}}$ to $\tilde{\mathcal{Q}}$ sending y, r_n^+ to y_0 and q_n^+ respectively, which is surjective by Proposition 5.4.8 and is a CQG morphism by Lemma 5.4.9. In other words, $(\tilde{\mathcal{S}}, \Delta_0, V)$ is indeed a universal object in $\mathbf{Q}'(D)$. It is clear that the maximal Woronowicz subalgebra of $\tilde{\mathcal{S}}$ for which the action is faithful, that is $QISO^+(D)$, is generated by $r_{n-1}^+ r_n^{+*}$, $n \geq 1$ and $r_0^+ y r_1^{+*}$, so again isomorphic with $C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$.

This is summarized in the following:

Theorem 5.4.11. *The universal object in the category $\mathbf{Q}'(D)$, that is $\widetilde{QISO^+(D)}$ exists and is isomorphic with $C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$. Moreover, the quantum group $QISO^+(D)$ is again isomorphic with $C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$.*

Remark 5.4.12. *This example shows that $QISO^+$ in general may not be matrix quantum group, that is may not have a finite dimensional fundamental unitary, even if the underlying spectral triple is of compact type. This is somewhat against the intuition derived from the classical situation, since for a classical compact Riemannian manifold the group of isometries is always a compact Lie group, hence has an embedding into the group of orthogonal matrices of some finite dimension.*

We end this chapter by noting that α gives an example where the quantum group of orientation preserving isometries does not have a C^* action. Before that, we recall some useful properties of the so called Toeplitz algebra from [26].

Proposition 5.4.13. *Let τ_1 be the unilateral shift operator on $l^2(\mathbb{N})$ defined by $\tau_1(e_n) = e_{n-1}$, $n \geq 1$, $\tau_1(e_0) = 0$. Then the Toeplitz algebra $C^*(\tau_1)$ is the C^* algebra generated by τ_1 , on $l^2(\mathbb{N})$. It contains all compact operators and moreover, the commutator of any two elements of $C^*(\tau_1)$ is compact.*

Let τ be the operator on \mathcal{H} defined by $\tau = \tau_1 \otimes \text{id}$.

Lemma 5.4.14. $B = \tau |B|$.

Proof : We note that

$$\begin{aligned} |B| \left(\begin{pmatrix} e_n \\ 0 \end{pmatrix} \right) &= (A - A^2 + cI)^{\frac{1}{2}} \left(\begin{pmatrix} e_n \\ 0 \end{pmatrix} \right) \\ &= \sqrt{\lambda_+ \mu^{2n} - \lambda_+^2 \mu^{4n} + c} \left(\begin{pmatrix} e_n \\ 0 \end{pmatrix} \right) = c_+(n)^{\frac{1}{2}} \begin{pmatrix} e_n \\ 0 \end{pmatrix} \end{aligned}$$

and hence $\tau |B| \left(\begin{pmatrix} e_n \\ 0 \end{pmatrix} \right) = c_+(n)^{\frac{1}{2}} \left(\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} \right) = B \left(\begin{pmatrix} e_n \\ 0 \end{pmatrix} \right)$.

Similarly, $\tau |B| \left(\begin{pmatrix} 0 \\ e_n \end{pmatrix} \right) = B \left(\begin{pmatrix} 0 \\ e_n \end{pmatrix} \right)$. This completes the proof of the Lemma.

□

Lemma 5.4.15.

$$\alpha(\tau) = \sum_{n \geq 1} \tau(P_n + Q_n) \otimes r_{n-1}^+ r_n^{+*},$$

where r_n^\pm are the elements of $\widetilde{QISO}^+(D)$ introduced before.

Proof : For all $n \geq 1$, we have

$$\begin{aligned} \alpha(\tau) \left(\begin{pmatrix} e_n \\ 0 \end{pmatrix} \right) &= \tilde{U}(\tau \otimes \text{id}) \tilde{U}^* \left(\begin{pmatrix} e_n \\ 0 \end{pmatrix} \right) \\ &= \frac{1}{2} \tilde{U}(\tau \otimes \text{id}) \tilde{U}^* \left[\begin{pmatrix} e_n \\ e_n \end{pmatrix} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \right] \\ &= \frac{1}{2} \tilde{U}(\tau \otimes \text{id}) \left[\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes r_n^{+*} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes r_n^{-*} \right] \\ &= \frac{1}{2} \tilde{U}(\tau \otimes \text{id}) \left[\begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes (r_n^{+*} + r_n^{-*}) + \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes (r_n^{+*} - r_n^{-*}) \right] \\ &= \frac{1}{2} \tilde{U} \left[\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} \otimes (r_n^{+*} + r_n^{-*}) + \begin{pmatrix} 0 \\ e_{n-1} \end{pmatrix} \otimes (r_n^{+*} - r_n^{-*}) \right] \\ &= \frac{1}{4} \tilde{U} \left[\begin{pmatrix} e_{n-1} \\ e_{n-1} \end{pmatrix} \otimes (2r_n^{+*}) + \begin{pmatrix} e_{n-1} \\ -e_{n-1} \end{pmatrix} \otimes (2r_n^{-*}) \right] \\ &= \frac{1}{2} \left[\begin{pmatrix} e_{n-1} \\ e_{n-1} \end{pmatrix} \otimes r_{n-1}^+ r_n^{+*} + \begin{pmatrix} e_{n-1} \\ -e_{n-1} \end{pmatrix} \otimes r_{n-1}^- r_n^{-*} \right] = \begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} \otimes r_{n-1}^+ r_n^{+*}. \end{aligned}$$

Similarly, $\alpha(\tau) \left(\begin{pmatrix} 0 \\ e_n \end{pmatrix} \right) = \begin{pmatrix} 0 \\ e_{n-1} \end{pmatrix} \otimes r_{n-1}^+ r_n^{+*}$ for all $n \geq 1$.

Moreover, $\alpha(\tau) \begin{pmatrix} e_0 \\ 0 \end{pmatrix} = \alpha(\tau) \begin{pmatrix} 0 \\ e_0 \end{pmatrix} = 0$.

Thus, $\alpha(\tau) = \sum_{n \geq 1} \tau P_n \otimes r_{n-1}^+ r_n^{+*} + \sum_{n \geq 1} \tau Q_n \otimes r_{n-1}^+ r_n^{+*} = \sum_{n \geq 1} \tau(P_n + Q_n) \otimes r_{n-1}^+ r_n^{+*}$ \square

Theorem 5.4.16. *The $*$ -homomorphism α is not a C^* action.*

Proof : We begin with the observation that each of the C^* algebras $\pi_{\pm}(S_{\mu,c}^2)$ is nothing but the Toeplitz algebra. For example, consider $\mathcal{C} := \pi_+(S_{\mu,c}^2)$. Clearly, $T = \pi_+(B)$ in an invertible operator with the polar decomposition given by, $T = \tau_1|T|$, hence τ_1 belongs to \mathcal{C} . Thus, \mathcal{C} contains the Toeplitz algebra $C^*(\tau_1)$, which by Proposition 5.4.13 contains all compact operators. In particular, $C^*(\tau_1)$ must contain $\pi_+(A)$ as well as all the eigenprojections P_n of $|\pi_+(B)|$ so it must contain the whole of \mathcal{C} . Similar arguments will work for $\pi_-(S_{\mu,c}^2)$.

Thus, $\tau = \tau_1 \oplus \tau_1 = \pi(B)|\pi(B)|^{-1}$ belongs to $\pi(S_{\mu,c}^2)$. If α is a C^* action, then for an arbitrary state ϕ on $QISO^+(D)$ we must have $\alpha_{\phi}(\tau) \equiv (\text{id} \otimes \phi) \circ \alpha(\tau)$ is in $\pi(S_{\mu,c}^2)$, hence $\alpha_{\phi}(\tau)P_+$ must belong to $\mathcal{C} = \pi_+(S_{\mu,c}^2)$, where P_+ denotes the projection onto \mathcal{H}_+ . By Proposition 5.4.13, this implies that $[\alpha_{\phi}(\tau)P_+, \tau_1]$ must be a compact operator. We claim that for suitably chosen ϕ , this compactness condition is violated, which will complete the proof of the theorem.

To this end, fix an irrational number θ and consider the sequence $\lambda_n = e^{2\pi i n \theta}$ of complex number of unit modulus. We note that the linear functionals which send the generator of $C(Z_2)$ (which is y) to 1 and the generator of the n -th copy of $C(\mathbb{T})$ (which is $r_{n-1}^+ r_n^{+*}$ by Proposition 5.4.8) to λ_n are evaluation maps and hence homomorphisms. Using Remark 1.1.6, we have a unital $*$ -homomorphism $\phi : QISO^+(D) = C(Z_2) * C(\mathbb{T})^{*\infty} \rightarrow \mathbb{C}$ which extends the above mentioned homomorphisms. Hence, $\alpha_{\phi}(\tau) = \sum_n \lambda_n \tau(P_n + Q_n)$. Moreover, we see that

$$\begin{aligned} & [\alpha_{\phi}(\tau)P_+, \tau_1] \begin{pmatrix} e_n \\ 0 \end{pmatrix} \\ &= (\text{id} \otimes \phi)\alpha(\tau) \begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} - \tau(\text{id} \otimes \phi) \left[\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} \otimes r_{n-1}^+ r_n^{+*} \right] \\ &= (\lambda_{n-1} - \lambda_n) \begin{pmatrix} e_{n-2} \\ 0 \end{pmatrix}, \quad n \geq 2. \end{aligned}$$

Similarly, $[\alpha_{\phi}(\tau)P_+, \tau_1] \begin{pmatrix} 0 \\ e_n \end{pmatrix} = (\lambda_{n-1} - \lambda_n) \begin{pmatrix} 0 \\ e_{n-2} \end{pmatrix}$. Hence, the above commutator cannot be compact since $\lambda_n - \lambda_{n-1}$ does not go to 0 as $n \rightarrow \infty$. \square

Corollary 5.4.17. *The subcategory of $\mathbf{Q}'(D)$ consisting of objects (\tilde{Q}, U) where α_U is a C^* action does not have a universal object.*

Proof: By Theorem 5.4.16, the proof will be complete if we can show that if a universal object exists for the subcategory (say \mathbf{Q}'_1) mentioned above, then it must be isomorphic with $\widetilde{QISO^+(D)}$. For this, consider the quantum subgroups \tilde{Q}_N , $N = 1, 2, \dots$, of $\widetilde{QISO^+(D)}$ generated by r_n^+ , $n = 1, \dots, N$ and y . Let $\pi_N : \widetilde{QISO^+(D)} \rightarrow \tilde{Q}_N$ be the CQG morphism given by $\pi_N(y) = y$, $\pi_N(r_n^+) = r_n^+$ for $n \leq N$ and $\pi_N(r_n^+) = 1$ for $n > N$.

We claim that $(\tilde{Q}_N, U_N := (\text{id} \otimes \pi_N) \circ V)$ is an object in \mathbf{Q}'_1 (where V denotes the unitary representation of $\widetilde{QISO^+(D)}$ on \mathcal{H}). To see this, we first note that for all N , $(\text{id} \otimes \pi_N)\alpha(A) = \sum_{n=0}^N AP_n \otimes \frac{1}{2\lambda_+} \{\lambda_+(1 + r_n^+ y r_n^{+*}) + \lambda_-(1 - r_n^+ y r_n^{+*})\} + \sum_{n=0}^N AQ_n \otimes \frac{1}{2\lambda_-} \{\lambda_+(1 - r_n^+ y r_n^{+*}) + \lambda_-(1 + r_n^+ y r_n^{+*})\} + \sum_{n=N+1}^\infty AP_n \otimes \frac{1}{2\lambda_+} \{\lambda_+(1 + y) + \lambda_-(1 - y)\} + \sum_{n=N+1}^\infty AQ_n \otimes \frac{1}{2\lambda_-} \{\lambda_+(1 - y) + \lambda_-(1 + y)\}$.

Among the four summands, the first two clearly belong to $\mathcal{A} \otimes \tilde{Q}_N$. Moreover, the sum of the third and the fourth summand equals $A(1 - \sum_{n=1}^N P_n) \otimes \frac{1}{2\lambda_+} \{\lambda_+(1 + y) + \lambda_-(1 - y)\} + A(1 - \sum_{n=1}^N Q_n) \otimes \frac{1}{2\lambda_-} \{\lambda_+(1 - y) + \lambda_-(1 + y)\}$ which is an element of $\mathcal{A} \otimes \tilde{Q}_N$.

We proceed similarly in the case of B , to note that it is enough to show that for all N ,

$$\begin{aligned} & \sum_{n=N+2}^\infty BP_n \otimes \frac{c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}}}{2c_+(n)} + \sum_{n=N+2}^\infty BP_n \otimes \frac{(c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})y}{2c_+(n)} \\ & + \sum_{n=N+2}^\infty BQ_n \otimes \frac{c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}}}{2c_-(n)} - \sum_{n=N+2}^\infty BQ_n \otimes \frac{(c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})y}{2c_-(n)} \end{aligned}$$

belongs to $\mathcal{A} \otimes \tilde{Q}_N$. The norm of the second and the fourth term can be easily seen to be finite. The first term equals $\frac{1}{2}B(1 - \sum_{n=1}^{N+1} P_n)[(A - A^2 + cI)^{-\frac{1}{2}} + (A - A^2 + cI)^{-1} \{ \frac{\lambda_-}{\lambda_+} A - (\frac{\lambda_-}{\lambda_+} A)^2 + cI \}^{\frac{1}{2}}] \otimes 1$ and therefore belongs to $\mathcal{A} \otimes \tilde{Q}_N$. The third term can be treated similarly.

Thus, there is surjective CQG morphism ψ_N from the universal object, say $\tilde{\mathcal{G}}$, of \mathbf{Q}'_1 to \tilde{Q}_N . Clearly, $(\tilde{Q}_N)_{N \geq 1}$ form an inductive system of objects in $\mathbf{Q}'(D)$, with the inductive limit being $\widetilde{QISO^+(D)}$, and the morphisms ψ_N induce a surjective CQG morphism (say ψ) from $\tilde{\mathcal{G}}$ to $\widetilde{QISO^+(D)}$. But $\tilde{\mathcal{G}}$ is an object in $\mathbf{Q}'(D)$, so must be a quantum subgroup of the universal object in this category, that is, $\widetilde{QISO^+(D)}$. This gives the CQG morphism from $\widetilde{QISO^+(D)}$ onto $\tilde{\mathcal{G}}$, which is obviously the inverse of ψ , and hence we get the desired isomorphism between $\tilde{\mathcal{G}}$ and $\widetilde{QISO^+(D)}$. \square

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